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AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

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THE DISTRIBUTION OF ELECTRONS IN THE VARIOUS ORBITS OF THE HYDROGEN ATOM

By H. C. UREY

ABSTRACT

Application of chemical thermodynamics to the equilibrium existing between a neutral atom, its ion, and electron gas. Equations are developed for the partial pressures of each of the atomic species present using the concepts of *partial molal free energy* and *fugacity*. It is shown that if the atoms in each of the resonated states are regarded as composing a perfect gas, the partial pressure of atoms having their electrons in any finite orbit would be zero, and the gas would be completely ionized at all temperatures and pressures. By assuming that atoms having electrons in orbits of high quantum numbers form imperfect gases due to the volume which the atoms themselves occupy, it has been possible to make a solution of this problem for monatomic hydrogen gas. The numerical calculations are made for 10,000° K., and pressures of 0.001 and 0.008 atmospheres. At this temperature and the former pressure only 0.0007 approximately of the total number of atoms have their electrons in higher orbits than the first. All atoms having their electrons in orbits of total quantum number 10 or less may be regarded as composing perfect gases.

I. INTRODUCTION

The application of chemical thermodynamics to the equilibria existing between gaseous ions and electrons has been made by Saha,[1] who has succeeded by this means in giving a very interesting explanation of the differences in spectra of stars of different temperatures, the differences in the spectra observed at various levels of the sun's atmosphere, and the apparent absence of certain elements in the sun which are well known on earth. Tolman[2] also has pub-

[1] Saha, *Philosophical Magazine*, **40**, 472, 809, 1920; **41**, 267, 1921; *Proceedings of the Royal Society*, A, **99**, 135, 1921.

[2] *Journal of American Chemical Society*, **43**, 1630–32, 1921.

lished a note, arriving independently at conclusions similar to those of Saha. Russell[1] has extended this work and has attempted to include in his considerations the common ion effect in mixtures of ionizable gases. These authors have considered reaction of the type of the dissociation of monatomic hydrogen into hydrogen ion and electron gas according to the equation

$$H = H^{+} + E^{-} .$$

In accordance with this reaction, at sufficiently low pressures and high temperatures, the gas becomes very highly dissociated. Under these conditions the number of atoms capable of emitting light is so small that we cannot recognize the light emitted by them. Therefore, the characteristic lines of the gas disappear. In this paper we shall investigate, more closely than has been done, the relative numbers of atoms existing in the various possible quantum states under conditions of thermodynamic equilibrium, with particular reference to monatomic hydrogen. This is a matter of some importance, since in previous discussions the number of hydrogen atoms in states between complete ionization and the lowest quantum state have been largely neglected.

II. THERMODYNAMIC EXPRESSION FOR PARTIAL PRESSURE OF ATOMS IN DIFFERENT QUANTIZED STATES

Let A_1, A_2, A_3, etc., represent any atom having one electron which may take up stable positions in the orbits 1, 2, 3, etc., respectively. The numbers are used to designate distinct states, and have no reference to the quantum numbers of the orbits. We will consider the equilibria of the reactions

$$A_1 = A_2,\ A_1 = A_3,\ \ldots\ldots,\ A = A_j,\ \ldots\ldots \tag{1}$$

and

$$A_1 = A^{+} + E^{-} . \tag{2}$$

This problem can be handled by well-known thermodynamic methods, provided we are willing to regard such atoms with their electrons in a particular quantum state as constituting a definite chemical species to which such concepts as partial molal free energy and fugacity are applicable.

[1] *Astrophysical Journal*, **55**, 119, 1922.

For any equilibrium we have the free energy equation

$$-RTlnK=\Delta F^\circ=\Delta H^\circ-T\Delta S^\circ \tag{3}$$

where K, the equilibrium constant, should for exactness be expressed in fugacities,[1] and ΔH° and ΔS° are the change in heat content and entropy when the reactions as written take place under standard conditions (i.e., the reactants disappear at unit fugacity and the products appear at unit fugacity).

Referring now to the reactions indicated by equation (1), it is evident, since the change is merely from one monatomic gas to another of the same atomic weight, that the entropy change ΔS° under standard conditions can be assumed zero. Hence, changing to the exponential form, equation (3) becomes

$$\frac{f_j}{f_1}=e^{-\frac{\Delta H_j^\circ}{RT}}. \tag{4}$$

Let us now define a function (s_j) such that $s_jp_j=f_j$; i.e., the product of s_j and the partial pressure is equal to the fugacity. Then

$$\frac{s_jp_j}{s_1p_1}=e^{-\frac{\Delta H_j^\circ}{RT}}, \qquad \text{or} \qquad \frac{p_j}{p_1}=\frac{s_1}{s_j}e^{-\frac{\Delta H_j^\circ}{RT}}. \tag{5}$$

Similarly, for the reaction represented by equation (2), the equilibrium may be expressed by

$$\frac{p_ip_e}{p_1}=\frac{s_1}{s_is_e}e^{-\frac{\Delta H_i^\circ}{RT}+\frac{\Delta S_i^\circ}{R}}, \tag{6}$$

where the subscripts i and e refer to the ion and electron, respectively. Since the molal fractions of hydrogen ion and electron must be the same, we may put $p_i=p_e$, and the value of ΔS° for this reaction is the entropy of the electron, if the difference in mass of the neutral atom and ion is neglected.

The sum of the partial pressures of various kinds of gas atoms is equal to the total pressure P, or

$$\sum p_j+p_i+p_e=P. \tag{7}$$

[1] See Lewis and Randall, *Thermodynamics and the Free Energy of Chemical Substances* (McGraw-Hill Book Co., 1923), chap. xvii, for a development of the notion of fugacity.

From equations (5) and (6) we get

$$p_j = p_1 \frac{s_1}{s_j} e^{-\frac{\Delta H_j^\circ}{RT}}, \tag{8}$$

and

$$p_i = p_e = \left[\frac{s_1}{s_i s_e} p_1 e^{-\frac{\Delta H_i^\circ}{RT} + \frac{\Delta S_i^\circ}{R}} \right]^{\frac{1}{2}}, \tag{9}$$

and substituting these in equation (7) we obtain

$$p_1 \sum \frac{s_1}{s_j} e^{-\frac{\Delta H_j^\circ}{RT}} + 2 \left[\frac{s_1}{s_i s_e} p_1 e^{-\frac{\Delta H_i^\circ}{RT} + \frac{\Delta S_i^\circ}{R}} \right]^{\frac{1}{2}} = P. \tag{10}$$

Solving this for p_1, we get

$$p_1 = \frac{P \sum \frac{s_1}{s_j} e^{-\frac{\Delta H_j^\circ}{RT}} + 2 \frac{s_1}{s_i s_e} e^{-\frac{\Delta H_i^\circ}{RT} + \frac{\Delta S_i^\circ}{S}}}{\left[\sum \frac{s_1}{s_j} e^{-\frac{\Delta H_j^\circ}{RT}} \right]^2} - \frac{2 \left[\sum \frac{s_1}{s_j} e^{-\frac{\Delta H_j^\circ}{RT}} \frac{s_1}{s_i s_e} e^{-\frac{\Delta H_i^\circ}{RT} + \frac{\Delta S_i^\circ}{R}} P + \left\{ \frac{s_1}{s_i s_e} e^{-\frac{\Delta H_i^\circ}{RT} + \frac{\Delta S_i^\circ}{R}} \right\}^2 \right]^{\frac{1}{2}}}{\left[\sum \frac{s_1}{s_j} e^{-\frac{\Delta H_j^\circ}{RT}} \right]^2} \tag{11}$$

and substituting this function in equation (8) we can solve for p_j. The sign preceding the second term of equation (11) is ambiguous. We select the negative sign, since, if P is set equal to $p_1 + 2p_i$, and $\sum \frac{s_1}{s_j} e^{-\frac{\Delta H_j^\circ}{RT}}$ equal to its first term 1, and $\frac{s_1}{s_i s_e} e^{-\frac{\Delta H_i^\circ}{RT} + \frac{\Delta S_i^\circ}{R}}$ equal to $\frac{p_i p_e}{p_1}$, this choice reduces equation (11) to an identity as it should. The term $e^{-\frac{\Delta H_j^\circ}{RT}}$ is finite, and if we extend the summation over an infinite number of quantum states without using the correction factor $\frac{s_1}{s_j}$, the summation would be infinite, and, therefore, the solution for p_1 impossible. We see that the function s_j must be of such a character as to change this series from a divergent to a convergent series.

III. DETERMINATION OF THE FUGACITIES OF THE DIFFERENT COMPONENTS

In order to evaluate the functions s_j, s_i, and s_e, it is necessary to know the equations of state of each of the molecular species present, and also the manner in which the mixture deviates from the perfect solution laws. These we do not know but can make plausible approximations, which will show qualitatively, at least, how we must proceed to solve the equations for p_j.

The ion and electron would be very imperfect gases if we had each pure and not mixed with particles of opposite charge. When present together, however, in a mixture, at low concentrations or pressures, we shall feel justified in assuming that they behave very nearly as perfect gases. We shall therefore take as the equations of state of the positive ion and electron

$$PV=RT \qquad \text{and} \qquad E=\tfrac{3}{2}RT+C. \tag{12}$$

This choice is the same as that used by Saha and others, and makes s_i and s_e equal to unity. The principal cause for the deviations from the law of the perfect gas in the case of the other atomic species may be assumed to be due to the volumes that the atoms themselves occupy. The volume occupied by atoms having the electron in a large orbit must be larger than the volume occupied by the atoms having the electrons in a smaller orbit. This was recognized by Bohr[1] in a qualitative way as the reason why the higher members of the Balmer series appear only at very low pressures, such as exist in the outer atmosphere of the stars. If we correct the gas law for the volume occupied by the molecules, the gas would follow the equations of state,

$$P(V-b_j)=RT, \qquad E=\tfrac{3}{2}RT+C, \tag{13}$$

in which b_j is four times the volume occupied by a mol of atoms having the electron in the j orbit.

As to the behaviors of such gases when mixed, Lewis and Randall have suggested that all gaseous mixtures may be regarded as perfect solutions, and that the fugacity of the gas in the mixture is equal to the fugacity of the gas at the same total pressure multiplied by

[1] Bohr, *Philosophical Magazine*, **26**, 9, 1913.

the mol fraction.[1] The fugacity of a gas obeying equation (12) is equal to the pressure and fugacity of a gas obeying equation (13), as is given by the relation[2]

$$lnf_j^P = ln\frac{RT}{V-b_j}+\frac{b_j}{V-b_j}, \quad \text{or} \quad f_j^P=\frac{RT}{V-b_j}e^{\frac{b_j}{V-b_j}}. \tag{14}$$

The symbol f_j^P refers to the fugacity of the pure gas A_j at the pressure P. This multiplied by the mol fraction will be the fugacity of the gas in the mixture. The partial pressure is the total pressure multiplied by the *mol* fraction and, therefore,

$$\frac{N_j}{\sum N_j+N_i+N_e}=\frac{p_j}{P}$$

and the fugacity of the gas in the mixture is

$$f_j=\frac{RT}{V-b_j}\frac{p_j}{P}e^{\frac{b_j}{V-b_j}}. \tag{15}$$

Since $\frac{RT}{V-b_j}=P$, this reduces to

$$f_j=p_je^{\frac{b_jP}{RT}} \tag{16}$$

and

$$s_j=e^{\frac{b_jP}{RT}}. \tag{17}$$

For $b_j=o_i$, $s_j=1$, and the fugacity becomes equal to the partial pressure.

It is of interest to note that s_j, as here defined, increases without limit as b_j becomes infinitely large, and hence, though the fugacity remains finite for the infinite orbit, the partial pressure of atoms in the infinite orbit becomes zero. The infinite series

$$\sum\frac{s_1}{s_j}e^{-\frac{\Delta H_j^\circ}{RT}}$$

is then convergent if $b_{j+1}-b_j$ becomes infinitely large and the summation is carried out over all states as described below. If b_j is proportional to any power of the quantum number higher than the first, then lim $b_{j+1}-b_j$ does become infinite and the series

[1] See Lewis and Randall, *op. cit.*, p. 226. [2] *Ibid.*, p. 196.

is convergent.[1] Other equations of state for these atomic species could undoubtedly be devised which would also make this series convergent, but our assumptions do show that equation (11) can be solved as would not be the case if $s_j = 1$ for all values of j. This explains why the lines of a spectral series become weaker as we go to the higher members but do not entirely disappear at any finite member.

IV. APPLICATION TO THE QUANTUM STATES OF MONATOMIC HYDROGEN

Let us now proceed to apply these considerations to a determination of the relative amounts of monatomic hydrogen in its different quantum states under conditions of such high temperature and low pressure that there is no appreciable amount of undissociated diatomic hydrogen left. We shall be able to simplify our problem by grouping together quantum states of the hydrogen atom which have the same total quantum number n and hence approximately the same energy.

The value of ΔH_j° is $Nh\nu_j$, where ν_j is the frequency of light emitted in a transition from the state of total quantum number n to the lowest quantum state. This is independent of the temperature since there is no change in heat capacity. For ΔH_i° and ΔS_i°, we take the same values as assumed by Saha and others, namely, for ΔH_i°, $Nh\nu_i + \frac{5}{2}RT$, where ν_i is the frequency of the head of the principal series, and the entropy of electron gas from the Sackur equation calculated at 1 atmosphere pressure and temperature T for ΔS_i°.

For the value of b_j we shall take four times the volume occupied by spheres having radii equal to the radii of the circular orbits or the half-major axes in the case of the elliptical orbits. It is not maintained that this will give accurate results, but the qualitative results, we believe, justify the attempt at a solution of the problem.

As for the number of distinct quantum states having the same total quantum number n, we may use the value $\frac{n(n+1)}{2}$. This

[1] A paper by Fowler, *Philosophical Magazine*, **45**, 1, 1923, has been published, since this paper was written, on "Dissociation Equilibria by the Method of Partitions," in which he called attention to the possibility of this series being divergent. We believe that the use of the fugacity function correctly interprets this apparent discrepancy.

can be obtained as is familiar from the theory of quantization in space.[1]

We are now ready to use equation (11). By grouping together the orbits as described above, and substituting for s_1 and s_j in accordance with equation (17), we can replace the summation occurring in equation (11) by

$$\sum \frac{n(n+1)}{2} e^{-\frac{Pb_n}{RT}} e^{-\frac{\Delta H^\circ_n}{RT}} .$$

We may now proceed to the calculations, and shall use the following values for the fundamental constants:

R_0, Rydberg's constant for hydrogen,	$109,677.7$	Birge[2]
h, Planck's constant,	6.5542×10^{-27}	Birge[3]
N, Avogadro's number,	6.059×10^{23}	
e, charge on the electron,	4.774×10^{-10} e.s.u.	
R, the gas constant,	8.316×10^{7} ergs per degree	
S, the entropy constant in the Sackur equation,	-2.63 cal/deg.[4]	

Then

$$b_n = \tfrac{16}{3}\pi n^6 (.532 \times 10^{-8})^3 N$$

and

$$\frac{b_n}{R} = 1.863 \times 10^{-2} n^6 .$$

If we make the calculation for 10,000° K., and 0.001 atmosphere, this gives

$$\frac{s_1}{s_n} = e^{(1-n^6)1.863 \times 10^{-9}} .$$

When the numerical values are substituted in the factor

$$e^{-\frac{NR_0h}{RT}\left(1-\frac{1}{n^2}\right)}$$

this becomes $10^{-6.8236\left(1-\frac{1}{n^2}\right)}$.

The calculation of all the terms of the summation $\sum \frac{s_1}{s_n} e^{-\frac{\Delta H_n}{RT}}$ would be very laborious, but an approximate method can be used

[1] See Sommerfeld, *Atombau* (3d ed.), chap. iv, paragraph 7 (Friedr. Vieweg und Sohn, Braunschweig, 1922). It might seem better, in accordance with the Bohr correspondence principle, to count positive and negative rotations separately, and use the value $n(n+1)$. This would obviously lead to no change in the ratios of the numbers of atoms in the different quantized states, but would decrease the entropy change accompanying ionization by the small term $R\ ln\ 2$.

[2] Birge, *Physical Review*, **21**, 589, 1921.

[3] *Ibid.*, N.S., **14**, 361, 1919.

[4] Lewis, *Physical Review*, **18**, 121, 1921.

which gives us sufficiently accurate results. The values of terms corresponding to different values of the total quantum number n are given in Table I, column 2, and these are plotted against the number of the orbit in Figure 1, curve II, and a smooth curve drawn through these points. The line should be discontinuous, as shown, for the first few orbits. The errors made on the rising part of the curve are partially balanced by errors in the falling part of the curve. We find from this that the area under this curve is 1.0007. Curve I

TABLE I

n	$\frac{s_1}{s_n}\frac{n}{2}(n+1)e^{-\frac{\Delta H^\circ_n}{RT}}$ $P=.001$ $T=10{,}000°$ K. $p_1=.183\times10^{-3}$	$\frac{s_1}{s_n}\frac{n}{2}(n+1)e^{-\frac{\Delta H^\circ_n}{RT}}$ $P=.008$ $T=10{,}000°$ K. $p_1=4.02\times10^{-3}$	$\frac{n}{2}(n+1)e^{-\frac{\Delta H^\circ_n}{RT}}$ Any total pressure $T=10{,}000°$ K. $p_1=0$
1	1.	1.	1.
2	2.288×10^{-5}	2.288×10^{-5}	2.288×10^{-5}
3	5.159×10^{-6}	5.159×10^{-6}	5.159×10^{-6}
4	4.008×10^{-6}	4.008×10^{-6}	4.008×10^{-6}
5	4.221×10^{-6}	4.221×10^{-6}	4.221×10^{-6}
6	4.877×10^{-6}	4.877×10^{-6}	4.877×10^{-6}
7	5.792×10^{-6}	5.792×10^{-6}	5.792×10^{-6}
8	6.907×10^{-6}	6.907×10^{-6}	6.907×10^{-6}
9	8.200×10^{-6}	8.200×10^{-6}	8.200×10^{-6}
10	9.519×10^{-6}	9.519×10^{-6}	9.643×10^{-6}
15	1.891×10^{-5}	1.596×10^{-5}	1.932×10^{-5}
20	2.915×10^{-5}	1.279×10^{-5}	3.279×10^{-5}
23	3.251×10^{-5}	4.837×10^{-6}	4.268×10^{-5}
25	3.194×10^{-5}	1.380×10^{-6}	5.003×10^{-5}
27	2.845×10^{-5}	....	5.798×10^{-5}
30	1.860×10^{-5}	....	7.102×10^{-5}
33	7.960×10^{-6}	....	8.545×10^{-5}
35	3.255×10^{-6}	....	9.548×10^{-5}

shows the course of the summation if the correction terms s_1/s_n are not used, and the calculated terms are given in column 4 of the table. Curve III shows a similar case to that shown in curve II but with P taken eight times as large. The data for this curve are given in column 3 of the table. These curves show the relative partial pressures, taking the pressure in the first orbit as unity.

Since we know that this summation is equal to 1, we can now solve with certainty equation (11) for the value of p_1. The value of the equilibrium constant for ionization, K_i, is $10^{-3.0402}$, i.e.,

$$\frac{s_1}{s_i s_e} e^{-\frac{\Delta H^\circ_i}{RT}+\frac{\Delta S^\circ_i}{R}} = 10^{-3.0402},$$

and this gives $p_1 = .183 \times 10^{-3}$. To secure the partial pressures of atoms in the other quantum states, it is only necessary to multiply the values given in the table by the value of p_1, as can readily be seen from equation (8). Thus, for p_2 we get 4.187×10^{-9}.

The interesting points to note in regard to this work are that the infinite series occurring in equation (11) is convergent, and that

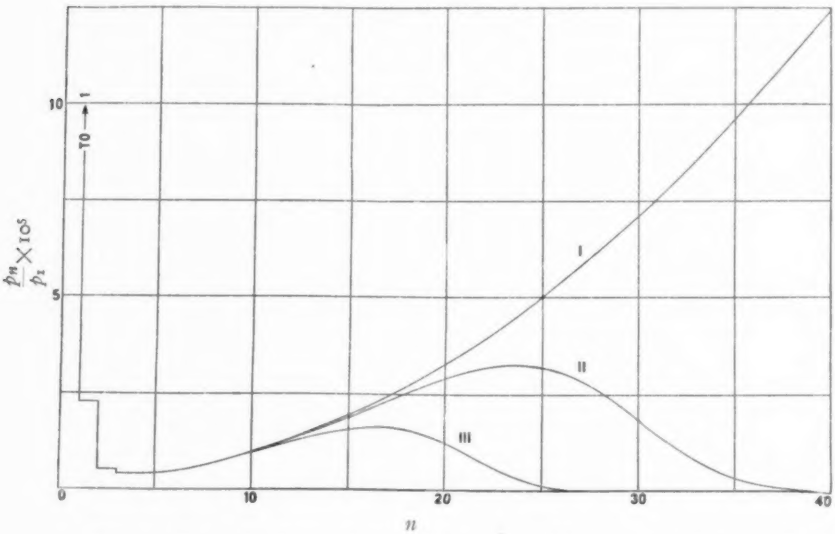

FIG. 1.—Curve I: Ratio of partial pressures with no correction for volume occupied by atoms. Curve II: Ratio of partial pressures at $T = 10{,}000°$ K. and $P = 0.001$ atmospheres. Curve III: Ratio of partial pressures at $T = 10{,}000°$ K. and $P = 0.008$ atmospheres.

for all calculations of this sort, in which we are interested, it may be taken as equal to unity. Moreover, the quantum states up to those of total quantum number 10 may be regarded as perfect gases at pressures below .01 atmosphere. This conclusion will also be true for other atoms than hydrogen.

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REMARKS ON THE LUMINOSITY AND DENSITY FUNCTIONS[1]

By FREDERICK H. SEARES

ABSTRACT

Comparison of observed and theoretical distribution of stars in space.—A combination of the luminosity and density functions of Kapteyn and Van Rhijn leads to the *apparent luminosity function*, which expresses the distribution of the absolute magnitudes of all stars brighter than apparent magnitude m_0. Values of this function were found for successive limits to $m_0 = 11.0$ by calculating the numbers of stars within the intervals $M \pm 1/2$ and $m \pm 1/2$ and summing the results with respect to m. Counts of the helium stars of Kapteyn and of the stars of other spectral types in the lists of spectroscopic parallaxes were made for similar intervals of M and m, and summed for comparison with the theoretical distribution. For the limit $m_0 = 5.0$, beyond which the observed data are incomplete, the agreement of frequencies is good for $M \gtreqless +1.5$. For $M < 0$, the calculated numbers are systematically in excess by an amount corresponding to about one magnitude in M, while for $M = 0$ to $+1.5$ there is a compensating excess in the observed numbers. The agreement is within the uncertainties affecting the data used. The observed distribution of absolute magnitudes for all spectral types together shows little trace of the giant and dwarf subdivisions appearing among the later types.

Systematic difference between mean and measured parallaxes of highly luminous stars.—Since the absolute magnitudes of intrinsically bright stars used in deriving the luminosity function proper depend on parallactic motions, the difference between the observed and calculated distribution of stars can be interpreted as a systematic difference between mean and measured parallaxes. This may arise in part from an extrapolation of the mean parallax formula; but the appearance of a similar difference in the case of Van Rhijn's revised mean parallaxes recently published suggests an error of more fundamental character. Strömberg's result, that the solar motion varies with the brightness of the reference stars, may afford a partial explanation. When reduced with the appropriate solar motion, the parallactic motions of giant stars will give mean parallaxes larger than those obtained with the constant solar velocity hitherto used, and hence a decrease in the brightness of the most luminous stars. The change is in the right direction, and apparently of the right order to account for much of the difference. The revision of mean parallaxes in accordance with Strömberg's result will lead to slight changes in the constants of both the luminosity and density functions.

Evidence on reliability of luminosity function afforded by the nearer stars.—A large percentage excess of low luminosities observed among the nearer stars has been cited as evidence that the mode M_0, or at least the dispersion, of the luminosity function is too small. Examination of the mean proper motions of stars at different distances tends to support this conclusion. Luyten's conclusion that M_0 must be increased, based on large proper motions, is invalidated by selection in the data discussed.

The luminosity and density functions are empirical formulae from which the total numbers of stars of each absolute magnitude in any portion of space can be calculated. Catalogues of observed absolute magnitudes, even when complete to a given limiting appar-

[1] *Contributions from the Mount Wilson Observatory*, No. 271.

ent magnitude, include only a small fraction of the stars within the region containing those actually observed; the more distant stars of low luminosity are either invisible or too faint for observation. An illustration will show how large is the percentage thus lost.

A catalogue, complete to apparent magnitude m_0, includes all stars satisfying the condition

$$M+5 \log \rho \gtreqless m_0 , \tag{1}$$

M being the absolute magnitude on Kapteyn's scale and ρ the distance in parsecs. The maximum distance involved is determined by the maximum luminosity occurring among the stars, which corresponds to about $M=-10$; thus, for the maximum,

$$\log \rho = 0.2\, m_0 + 2 . \tag{2}$$

For $m_0=5.0$, maximum $\rho=1000$ parsecs. The number of stars brighter than $m_0=5.0$ is about 1500. Hence, a catalogue complete to the fifth apparent magnitude would contain 1500 stars, scattered throughout a sphere of 1000 parsecs radius. But the total number of stars within this sphere, as indicated by the density and luminosity functions of Kapteyn and Van Rhijn,[1] is of the order of 30,000,000.

This disparity in numbers also appears for other values of m_0, and suggests the difficulty of obtaining reliable information on the distribution of stars in space. It also shows that the theoretical distribution is not directly comparable with the results of observation, and, at the same time, emphasizes the importance of such a comparison, which is of interest for other reasons as well. Thus the luminosity and density functions depend largely on a statistical discussion of parallactic motions, particularly of stars much fainter than those occurring in catalogues of individually determined absolute magnitudes. Are these functions consistent with what we know of the luminosity and distribution of stars of the brighter apparent magnitudes? Further, is the adopted luminosity function compatible with the giant and dwarf subdivisions of Russell's diagram?

[1] *Ibid.*, Nos. 188, 229; *Astrophysical Journal*, **52**, 23, 1920; **55**, 242, 1922.

The comparison in question is easily made with the aid of what may be called the apparent luminosity function. This function, which expresses the frequencies of absolute magnitudes among the stars brighter than m_0, can be derived from the density function and the luminosity function proper. Its values are directly comparable with observational results, and should correspond to the sums of the frequencies for the different spectral types shown by the diagram of Russell with its subdivision into giants and dwarfs.

The foregoing illustration shows the apparent luminosity function to be concerned mainly with the most luminous stars in a certain spherical region. Since the dispersion in absolute magnitude is known to be large, the modal M of the apparent distribution must lie far above that of the true luminosity function. As the limit m_0 is increased, the mode of the apparent function approaches the true mode, but for any practicable limit the difference is always large. It is this circumstance which makes it so difficult to obtain information about the descending branch of the true luminosity function.

FORMULAE FOR THE APPARENT LUMINOSITY FUNCTION

The apparent luminosity function can be expressed analytically; but, since we are concerned chiefly with numerical values, it is simpler to deal with a function $\psi(M, m)$, which gives directly the number of stars included within the limits $M \pm 1/2$ and $m \pm 1/2$. The summation of the numerical values of $\psi(M, m)$ for $m = 0.5$, $1.5, 2.5 \ldots.$ will then give the distribution of the absolute magnitudes of all stars brighter than m_0, which is the required numerical expression of the apparent luminosity function.

Since

$$\log \rho = 0.2\,(m - M) \tag{3}$$

it follows that $\psi(M, m)$ will be the number of stars in a spherical shell whose thickness is determined by the condition that m and M in (3) must lie within the limits $m \pm 1/2$ and $M \pm 1/2$.

The luminosity function

$$\phi(M) = e^{p + qM + rM^2} \tag{4}$$

gives the number of stars in the interval $M \pm 1/2$, expressed as a fraction of the total number of stars. The density function

$$\Delta(\rho) = e^{h+k \log \rho + l(\log \rho)^2} \tag{5}$$

gives the number of stars in a volume of one cubic parsec, situated at the distance ρ from the center of the system. Hence, the required number of stars in the spherical shell is

$$\psi(M, m) = 4\pi\rho^2 d\rho \Delta(\rho)\phi(M).$$

Theoretically, (5) is defective, in that for $\rho = 0$ it gives $\Delta(\rho) = 0$, whereas the density actually is a maximum at this point. Practically, the defect is usually of little consequence, for the maximum of (5) occurs at about $\rho = 60$ parsecs ($\log \rho = 1.8$), so that only a small region near the center of the system is affected. But in what follows the nearby stars are important, and it is convenient to use $\Delta(\rho) = A = \text{const.}$ for $\log \rho \lesseqgtr 1.6$, and equation (5) only when $\log \rho > 1.6$. Hence

$$\psi_1(M, m) = 4\pi\rho^2 d\rho A \phi(M) \qquad (\rho = 0 \text{ to } \log \rho = 1.6) \tag{6}$$

$$\psi_2(M, m) = 4\pi\rho^2 d\rho \Delta(\rho)\phi(M) \qquad (\log \rho = 1.6 \text{ to } +\infty) \tag{7}$$

From (3)

$$\rho = e^{a(m-M)} \qquad a = 0.2/\text{Mod.}$$

$$d\rho = a e^{a(m-M)} dm = -a e^{a(m-M)} dM$$

which determines the thickness of the shell, since dm and dM for the maximum range in m and M are both unity. Hence, for substitution into (6) and (7),

$$\rho^2 d\rho = a e^{3a(m-M)}$$

whence

$$\psi_1(M, m) = 4\pi a \phi(M) e^{3a(m-M)} A \qquad (m - M = -\infty \text{ to } +8) \tag{8}$$

$$\psi_2(M, m) = 4\pi a \phi(M) e^{3a(m-M)} \Delta(\rho) \qquad (m - M = +8 \text{ to } +\infty) \tag{9}$$

$$\log \Delta(\rho) = h + 0.2\, k(m-M) + 0.04\, l(m-M)^2 \tag{10}$$

The thickness of the shell $d\rho$ has been calculated from a differential formula. The values of both $d\rho$ and dM are finite, however, and, since the median value of ρ does not correspond exactly to the median values of m and M, results computed by (8) and (9) must

be multiplied by a factor differing slightly from unity to obtain correct results. Allowance for this is made later.

The constants h, k, and l of the density function vary with the galactic latitude. To avoid a direct calculation of these quantities for the sky as a whole, to which the proposed comparison refers, the results given by Kapteyn and Van Rhijn for latitude 30°,

$$h = -3.602, \qquad k = +2.381, \qquad l = -0.655,$$

have been used.[1] This was possible since the change in $\log N_m$ (N_m = total number of stars brighter than m) is much the same for latitude 30° as for the whole sky. The nearly constant difference in the two series of logarithms affects only h to any appreciable extent, and is easily taken into account later.

Equations (8) and (9) thus take the form

$$\log \psi_1(M, m) = \log \phi(M) - 0.584 + 0.6(m-M) \qquad (m-M = -\infty \text{ to } +8) \quad (11)$$

$$\log \psi_2(M, m) = \log \phi(M) - 2.840 + 1.076(m-M) - 0.0262(m-M)^2 \qquad (m-M = +8 \text{ to } +\infty) \quad (12)$$

where $\psi(M, m)$ represents the total number of stars in the sky having magnitudes between the limits $M \pm 1/2$ and $m \pm 1/2$, on the assumption that the density function has everywhere the values corresponding to galactic latitude 30°. To allow for the neglected factor, and to refer the results to the mean density function for all latitudes, 0.089, as shown below, must be added to the right members. The values of $\log \phi(M)$ are given in Table I*a*, *Mount Wilson Contributions* No. 229. The zero point for M corresponds to $M = m - 5 \log \rho$.

Values of $\psi(M, m)$ were calculated by (11) and (12) for all combinations of $m = 0.5, 1.5 \ldots\ldots 10.5$ and $M = -10.5, -9.5 \ldots\ldots +6.5$. The results for each m were then summed, thus giving the total numbers of stars, of all absolute magnitudes, within the limits $m \pm 1/2$. These sums should agree closely with the observed values of $N_{m \pm 1/2}$. For comparison with the observed data in *Groningen Publication*, No. 27, the totals for $m = 3.5, 4.5 \ldots\ldots 10.5$,

[1] *Mt. Wilson Contr.*, No. 229, p. 7; *Astrophysical Journal*, **55**, 248, 1922.

were themselves summed to obtain calculated values of N_m for the limits $m=4.0, 5.0 \ldots 11.0$.

The corresponding values of log N_m are in the second column of Table I; the residuals O−C, referred to values for 30° in Table V, *Groningen Publication*, No. 27, after reduction to the area of the whole sky, are in the fourth column. The differences are systematic because of the neglected factor arising from the use of differential formulae over finite intervals, but nearly enough constant to check the calculation. The large differences for $m=4.0$ and 5.0 are only those usually appearing in any attempt to include

TABLE I

Observed and Calculated Values of Log N_m

M	log N_m, 30°			log N_m, 0°–90°		
	Cal.	Obs.	O−C	Obs.	$O-C_1$	$O-C_2$
4.0........	2.526	2.637	+0.111	2.648	(+0.122)	(+0.033)
5.0........	3.070	3.140	+0.070	3.165	+0.095	+0.006
6.0........	3.598	3.634	+0.036	3.674	+0.076	−0.013
7.0........	4.104	4.120	+0.016	4.175	+0.071	−0.018
8.0........	4.589	4.598	+0.009	4.664	+0.075	−0.014
9.0........	5.049	5.067	+0.018	5.143	+0.094	+0.005
10.0........	5.484	5.490	+0.006	5.579	+0.095	+0.006
11.0........	5.892	5.905	+0.013	6.010	+0.118	+0.029
Mean					0.089	

the bright stars in the representation by the density and luminosity functions.

The fifth column of the table gives observed values of log N_m for all galactic latitudes, also derived from *Groningen Publication*, No. 27. The deviations of these from the calculated values for latitude 30°, $O-C_1$, are also nearly constant, which shows that the density function for 30° can really be used as indicated for the stars as a whole. The mean difference (the first being omitted) is 0.089, which includes the effect of both the neglected constants referred to above. The multiplication of the values of $\psi(M, m)$ already found by 1.227 and a repetition of the summation should then represent the stars for all galactic latitudes with residuals in log N_m as shown in the last column of Table I.

The detailed results are shown in Tables II and III, where, first of all, it is to be noted that the international zero point for M has now been introduced. For convenience, the median and limiting values of log ρ have been entered in the last columns of these tables. The course of the zigzag rules shows the numbers associated with any given value of log ρ. In Table II, for example, the number 1213 stands opposite $M=+2.5$ and $m=6.5$, and belongs with the sequence corresponding to log $\rho=1.8$, thus indicating that 1213 stars between absolute magnitudes $+2.0$ and $+3.0$ and apparent

TABLE II

VALUES OF $\psi(M, m)$ NUMBERS OF STARS IN THE INTERVAL $M\pm\frac{1}{2}$ AND $m\pm\frac{1}{2}$

M	m										Median log ρ
	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5	
−5.5	1	1	2	5	10	15	21	27	31	31	4.2
4.5	1	3	8	18	37	68	111	160	205	232	4.0
3.5	2	6	19	49	115	238	438	713	1028	1315	3.8
2.5	3	10	34	102	269	630	1303	2393	3899	5623	3.6
1.5	4	13	50	168	499	1315	3076	6368	11690	19050	3.4
−0.5	4	16	50	189	635	1892	4989	11670	24150	44360	3.2
+0.5	3	13	53	170	643	2158	6427	16940	39630	82040	3.0
1.5	2	10	38	153	492	1862	6252	18620	49090	114800	2.8
2.5	1	6	24	95	377	1213	4592	15420	45920	121100	2.6
3.5	1	3	13	50	200	794	2559	9683	32510	96830	2.4
4.5		1	6	23	90	359	1429	4603	17420	58480	2.2
5.5		1	2	9	35	138	551	2193	7063	26730	2.0
6.5			1	3	11	46	181	721	2871	9247	1.8
7.5				1	3	13	51	202	804	3199	1.6
8.5					1	3	12	48	192	766	1.4
9.5						1	2	10	39	156	1.2
10.5							1	2	7	27	1.0
+11.5									1	4	0.8
$N_{m\pm\frac{1}{2}}$	22	83	300	1035	3417	10745	31995	89774	236551	583991	
$N_{m+\frac{1}{2}}$	29	112	412	1447	4864	15609	47604	137377	373927	957917	
Mode				−0.39	+0.04	+0.48	+0.91	+1.34	+1.77	+2.20	

magnitudes 6.0 and 7.0 are situated in the shell whose median log ρ is 1.8. This shell is the first for which the density enters in the exponential form, equation (11) being used for all the inner shells.

In Table III the values of log ρ correspond to the radii of the spheres having the stellar content indicated by diagonal sequences of numbers. Thus the sequence 4, 18, 71 . . . . 13,081 represents the stars brighter than $m=11.0$ within the sphere log $\rho=1.9$. For log $\rho=0.9$ the sequence is so nearly complete that the total may

be estimated at 106, with an uncertainty of one or two stars. This total must agree with the value of $4\pi\rho^3 A/3$, where A is the number of stars per cubic parsec near the sun. The value of A found by Kapteyn and Van Rhijn is 0.045, but the effective value used above is 0.051. Hence $4\pi\rho^3 A/3=107$.

TABLE III

CALCULATED NUMBERS OF STARS IN THE INTERVAL $M\pm\frac{1}{2}$ BRIGHTER THAN m

M	m										Limiting log ρ
	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	
−5.5	1	2	3	9	19	34	55	82	113	144	4.3
4.5	1	4	12	30	67	135	246	406	611	843	4.1
3.5	2	8	27	76	191	429	867	1580	2608	3923	3.9
2.5	4	14	48	150	419	1049	2352	4745	8644	14267	3.7
1.5	5	18	68	236	735	2050	5126	11494	23184	42234	3.5
−0.5	5	21	71	260	895	2787	7776	19446	43596	87956	3.3
+0.5	4	17	70	240	883	3041	9468	26408	66038	148078	3.1
1.5	3	13	51	204	696	2558	8810	27430	76520	191320	2.9
2.5	2	8	32	127	504	1717	6309	21729	67649	188749	2.7
3.5	1	4	17	67	267	1061	3620	13303	45813	142643	2.5
4.5	1	2	8	31	121	480	1909	6512	23932	82412	2.3
5.5		1	3	12	47	185	736	2929	9992	36722	2.1
6.5			1	4	15	61	242	963	3834	13081	1.9
7.5				1	4	17	68	270	1074	4273	1.7
8.5					1	4	16	64	256	1022	1.5
9.5						1	3	13	52	208	1.3
10.5								2	9	36	1.1
+11.5									1	5	0.9
N_m	29	112	412	1447	4864	15609	47603	137376	373926	957916	
log N_m	1.462	2.049	2.615	3.160	3.687	4.193	4.678	5.138	5.573	5.981	
O−C	+0.133	+0.074	+0.033	+0.005	−0.013	−0.018	−0.014	+0.005	+0.006	+0.029	
Mode				−0.5	−0.1	+0.4	+0.8	+1.2	+1.6	+1.9	

As a further control, the comparison of the sums at the bottom of Table III with the corresponding values of N_m from *Groningen Publication*, No. 27, reproduces the residuals in the last column of Table I. It need scarcely be remarked that only the first two or three figures of the numbers in Tables II and III are of significance.

Disregarding the theoretical defect of (5), we can integrate (9) with respect to M, m being constant, and obtain the total number of stars in the interval $m\pm\frac{1}{2}$. The result can be expressed in the form

$$N_{m\pm\frac{1}{2}}=A_1\frac{H_1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-H_1^2(M-M_0)^2}dM=A_1 \qquad (13)$$

where

$$\left.\begin{array}{l}\log A_1 = +0.05344 + 0.6918\,m - 0.01489\,m^2 \\ M_1 = -7.33 + 0.4315\,m \qquad H_1 = +0.37385\end{array}\right\} \qquad (14)$$

The frequency function of the absolute magnitudes of all stars in the interval $m \pm 1/2$ is therefore an error curve having the mode M_1. The modification of the density function for values of $\log \rho < 1.6$ introduced above would disturb but slightly the symmetry of the distribution expressed by (13). The frequency function for N_m, on the other hand, including all stars brighter than $m = m_0$ is a skew curve, for it is the sum of the series of error functions obtained by introducing successive values of m into (13) and (14). The modal values for both $N_{m\pm\frac{1}{2}}$ and N_m are given in the last lines of Tables II and III. With increasing m, the mode for the apparent luminosity function (Table III) increases nearly as fast as that for the stars in the interval $m \pm 1/2$, but even for $m_0 = 11.0$ is nearly 6 magnitudes above the mode $M_0 = +7.7$ adopted for the true luminosity function.

COMPARISON OF APPARENT LUMINOSITY FUNCTION WITH OBSERVED DATA

Individual stars have not yet been observed with the completeness necessary for a full comparison with the calculated distribution shown in Table III. The faint stars are still largely missing. The most homogeneous material available is the list of 1646 spectroscopic parallaxes,[1] supplemented by the parallaxes of helium stars derived by Kapteyn[2] and of the A stars by Adams and Joy.[3] Counts of the B and F—M stars for intervals $M \pm 1/2$ to successive limits m_0 are given in Table IV. Even for the areas covered, the numbers are complete only to $m_0 = 5.0$, and below this limit the data cannot be used for a general comparison because of an unknown degree of selection favoring stars of low luminosity.

From *Harvard Annals*, **64**, 91, it appears that to $m = 5.0$ the relative numbers of stars of various spectral groups are represented

[1] *Mt. Wilson Contr.*, No. 199; *Astrophysical Journal*, **53**, 13, 1921.

[2] *Mt. Wilson Contr.*, Nos. 82, 147; *Astrophysical Journal*, **40**, 43, 1914; **47**, 104, 1918.

[3] *Mt. Wilson Contr.*, No. 244; *Astrophysical Journal*, **56**, 242, 1922.

by the fractions in the second column of Table V. From Table III the total number to this limit is 1447. Hence, for the whole sky, the distribution among the spectral groups is that shown in the third column of Table V. The corresponding totals actually

TABLE IV

OBSERVED NUMBERS OF STARS IN THE INTERVAL $M \pm \frac{1}{2}$ BRIGHTER THAN m

M	m(B Stars)			m(F, G, K and M Stars)					
	5.0	6.0	7.0	5.0	6.0	7.0	8.0	9.0	10.0
−5.5	1	1	1						
4.5	3	3	3	3	3	3	3	3	3
3.5	15	15	15	6	8	8	8	8	8
2.5	24	28	28	14	17	19	22	22	22
1.5	63	84	85	17	25	30	33	34	34
−0.5	55	113	117	52	92	104	104	105	106
+0.5	39	97	107	158	324	443	463	466	467
1.5	10	48	56	77	158	215	233	248	250
2.5		13	18	37	80	111	133	150	152
3.5		2	4	33	93	134	165	183	186
4.5				13	42	79	111	128	131
5.5				4	17	53	90	120	125
6.5				4	10	24	51	86	93
7.5					1	4	9	21	23
8.5					1	3	6	18	21
9.5								4	12
10.5							1	6	11
N_m	210	404	434	418	871	1230	1432	1602	1644

TABLE V

NUMBERS OF STARS TO $m = 5.0$

Spectrum	Fraction	No.	Obs.	Factor
B0–B9	0.24	347	210	1.65
A0–A9	0.21	304	....	....
F0–M	0.55	796	418	1.90
Total	1.00	1447		

observed, by Table IV, are 210 B's and 418 of types F, G, K, and M.[1] The required factors for reduction to the whole sky are therefore 1.65 and 1.90, respectively. Hence, the "observed" distribution for all types except A is found by multiplying the

[1] About twenty A stars are also included in the number used for types F to M.

numbers in the second and fifth columns of Table IV by these factors and adding the two series of products. The results are in the third column of Table VI.

The spectroscopic parallaxes of the A stars depend on the assumption that for a given spectrum the absolute magnitude has one of two possible values, corresponding to sharp and diffuse spectral lines. The dispersion in luminosity is so small that this method gives excellent results, but the detailed distribution over the narrow interval within which the absolute magnitudes fall

TABLE VI

REPRESENTATION OF STARS TO $m=5.0$

M	Cal.	B+F to M	A	Obs.	O−C
−5.5	9	2		2	− 7
4.5	30	11		11	− 19
3.5	76	36		36	− 40
2.5	150	67		67	− 83
1.5	236	136	3	139	− 97
−0.5	260	190	38	228	− 32
+0.5	240	364	122	486	+246
1.5	204	163	111	274	+ 70
2.5	127	70	28	98	− 29
3.5	67	63	2	65	− 2
4.5	31	25		25	− 6
5.5	12	8		8	− 4
6.5	4	8		8	+ 4
+7.5	1				− 1
Total	1447	1143	304	1447	0

remains undetermined. The data lead, however, very directly to a value for the mode of the luminosity-curve, for they include many cluster stars, whose mean absolute magnitude is itself the required mode. This last statement presupposes a symmetrical distribution of luminosity, and, further, that all the A stars in one or more clusters have been completely observed. For certain groups like the Pleiades and the Taurus Stream the latter condition, at least, is nearly enough satisfied. The result for the mode is $M_0=+2.0$ (-3.0, Kapteyn's scale). For the modulus of dispersion Kapteyn[1] has found $H=0.80$, whence the probable dispersion is $0.477/0.80=0.60$, and

$$\phi_A(M)=e^{-0.64(M+3.0)^2} .$$

[1] *Mt. Wilson Contr.*, No. 147, p. 72; *Astrophysical Journal*, **47**, 262, 1918.

It is doubtful if the A stars really have an error-curve distribution, but the assumption affords a convenient means of smoothing the data, which is admissible since the adopted dispersion is of the order of the difference in absolute magnitude found by Adams and Joy for stars having sharp and diffuse lines.

To calculate the apparent distribution of the A stars, it is sufficient to use $\Delta(\rho) = \text{const.}$ Equation (11) then gives

$$\log \psi_A(M, m) = -0.278(M+3.0)^2 + 0.6(m-M) + \text{const.} \qquad (15)$$

The results for the 304 stars brighter than $m_0 = 5.0$ are in the fourth column of Table VI, the modal M being $+0.9$ on the international scale. This value of the mode may be compared with $+1.4$ obtained from Russell's diagram by extending the line of maximum frequency of the dwarfs until it joins that of the B's. The data used for the latter value, Figure 1, *Mount Wilson Contribution*, No. 226,[1] include many stars fainter that $m = 5.0$, which will account for at least part of the difference of 0.5 mag.

The inclusion of the A stars with the other spectral types and a comparison of the totals with the calculated distribution in the second column of Table VI (transcribed from Table III) lead to the differences in the last column of Table VI. This final result of the comparison is also shown in Figure 1. Superficially, the agreement is bad. There is a marked systematic excess in the calculated numbers of stars of high luminosity, while in the interval $M = 0$ to $+1.5$, the observed numbers are greatly in excess. More critically examined, however, the agreement turns out better than might have been expected, for several disturbing factors may enter:

1. The luminosity function is a statistical formula involving stars much fainter, in general, than those used above. It is based on the mean parallax formula, in which two of the three constants were derived exclusively from parallactic motions of stars of the fourth to the eleventh magnitudes.[2] The comparison, therefore, involves an extrapolation. Further, it is based on less than 1500

[1] *Astrophysical Journal*, **55**, 165, 1922. For a detailed plot showing individual stars see *Annual Report of the Director of the Mt. Wilson Observatory, Year Book of the Carnegie Institution of Washington*, 1921, p. 270.

[2] *Mt. Wilson Contr.*, No. 188, p. 4; *Astrophysical Journal*, **52**, 26, 1920.

stars, part of which are known to deviate more or less from the general statistical relations applying to the stars at large.

2. Any uncertainty in the constants of the luminosity function affects both the mode and the dispersion of the apparent distribution. Let M_0 and R be the mode and the probable dispersion of the luminosity function, and M_1 and R_1 the corresponding constants for the distribution function of the stars in the interval $m \pm 1/2$ (see equation [13]). Variations δM_0 and δR lead to the changes δM_1 and δR_1 defined by

$$\left.\begin{aligned} \delta M_1 &= \delta M_0 - 11.9\,\delta R + 0.51\,m\delta R \\ \delta R_1 &= +0.32\,\delta R \end{aligned}\right\} \qquad (16)$$

These relations, derived from the formulae of *Contribution* No. 188, p. 12, take into account the changes in the density function necessarily resulting from any modification of the luminosity function. A change in M_0 changes the mode of the apparent distribution function by the same amount. Further, M_1 is especially sensitive to changes in R. Thus for $m = 5$, an error of 0.1 mag. in the probable dispersion, unless compensated by an error in M_0, would affect M_1 by about 1 mag. R_1, however, is not much influenced by errors in R. These statements apply to the stars in the interval $m \pm 1/2$, but with minor modifications they are equally true of all stars brighter than m_0.

3. The constants of the luminosity function seemingly require some revision because of Strömberg's result[1] that the sun's motion relative to dwarf stars is larger than its motion relative to giants. Since the percentage of dwarfs increases with m, parallactic motions must eventually be reduced with a solar velocity which increases with increasing m, instead of the constant velocity hitherto used.

4. Any systematic difference between measured parallaxes and mean parallaxes must influence the comparison, for the calculated distribution depends largely on mean parallaxes. The revised values of the mean parallaxes recently published by Van Rhijn[2] present such a difference. It affects stars brighter than $M = 0$, which have very small proper motions, and corresponds to about

[1] *Mt. Wilson Contr.*, No. 245; *Astrophysical Journal*, **56**, 265, 1922.

[2] *Groningen Publication*, No. 34, 1923.

1 magnitude, with the same algebraic sign as the difference revealed by Figure 1. The mean parallaxes used for the derivation of the luminosity curve differ little from Van Rhijn's revised values, and hence must present a similar deviation from measured parallaxes, larger, if anything, than that indicated by Van Rhijn's discussion.

This seems to be the principal source of the systematic difference for the luminous stars shown by Figure 1. Its origin requires further investigation. Van Rhijn believes that the difficulty lies with the measured parallaxes, that their systematic errors are large enough to have an important percentage effect on stars as distant

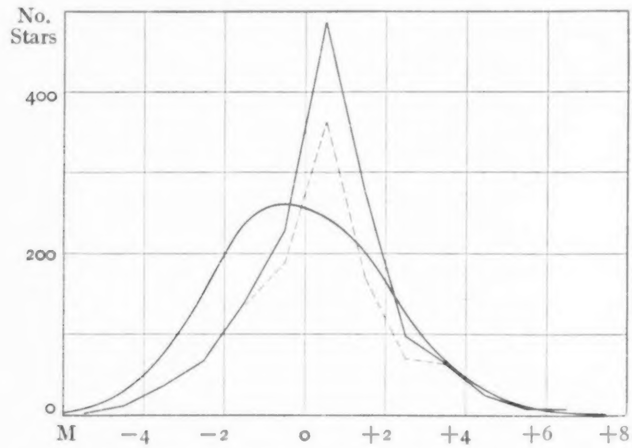

FIG. 1.—The curve indicates the numbers of absolute magnitudes among the stars brighter than the fifth apparent magnitude, calculated from the luminosity and density functions. The full line represents observed numbers. The area between the dotted and full lines shows the adopted distribution of the A stars.

as those involved. But his argument in favor of mean parallaxes tacitly assumes the sun's motion to be independent of the luminosity of the reference stars. Strömberg's results show, however, that this is not the case. The inclusion of this circumstance should remove a considerable portion of the outstanding difference.

5. Finally, though a very minor point, the very luminous A stars having *c* and *ac* characteristics have been classified with much fainter stars. The method of Adams and Joy does not apply to these objects, but they have been treated here as though it did.

This contributes to the difference shown by Figure 1, but the effect is slight because the stars are few in number.

In view of these circumstances the agreement of calculation with observation is probably all that can be expected at present. In fact, the comparison subjects the luminosity and density functions to a rather severe test, for a systematic error of 0".003 or so in the parallaxes of distant stars will account for practically all the discordance.

Since the comparison does not extend to stars fainter than $m = 5.0$, the compatibility of the luminosity function with the giant and dwarf subdivisions is not fully tested. Nevertheless, the numbers in Table IV for the fainter apparent magnitudes indicate that the summation over all spectral types tends to wipe out the double maximum shown by the individual frequency curves for the later types. Although less than 2 per cent of the stars brighter than $m = 10.0$ have been observed, the double maximum is scarcely in evidence, and the inclusion of the remaining stars will probably remove it altogether.

REPRESENTATION OF STARS NEAR THE SUN

The numbers in Tables III and IV indicate that in the part of the sky covered most of the stars of low luminosity which are apparently brighter than 10.0 have already been observed. Thus for spheres of different size the calculated and "observed" numbers are:

$\log \rho$	0.5	0.7	0.9	1.1	1.3	1.5
Observed No.	12	31	88	235	549	1086
Calculated No.	7	27	107	427	1698	6761

The ratios, observed to calculated, fall off rapidly, but all the nearer stars are seemingly either known or accounted for. Of the 107 stars within 8 parsecs, 80 per cent are found in the "observed" list. On the other hand, the total number of stars between $m = 9.0$ and 10.0 exceeds 200,000, of which only 42 appear in the spectroscopic list. This enormous number of objects of unknown distance and, for the most part, of unknown proper motion, suggests that the nearer stars of low luminosity may have been less completely observed than the tabulation indicates. If so, the readjustment of

theory to observation will eventually require a considerable increase in both the mode and the dispersion of the luminosity curve. Eddington's well-known list of 19 stars[1] has been cited as affording evidence in this direction, and, likewise, the discovery in recent years of numerous instances of large proper motion among the fainter apparent magnitudes. Further, Luyten,[2] from a study of large proper motions, concludes that the luminosity function will probably be modified in the manner indicated.

TABLE VII

DISTRIBUTION OF STARS NEAR THE SUN

Log ρ		m														Totals
		<2.0	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5	11.5	12.5	13.5	14.5	
$\overline{\leq}$0.5	Cal. . .	1	1	1	1	1	1	1								7
	Sp.-C.	+3	−1	+3	−1	+1	−1	+1								+5
	L.-C. .	+2	−1	+1	−1	−1	−1	0		+2	+1		+1			+3
	$\bar{\mu}$*					3″.2	3″.4	3″.5	3″.7	3″.8	3″.9					
0.5–0.7	Cal. . . .	2	1	2	3	3	3	2	2	1	1					20
	Sp.-C.	0	−1	−2	−1	−1	+1	−2	+7	−1	−1					−1
	L.-C. .	−1	−1	−2	−2	−2	−1	−1	+2	+2	−1		+1			−6
	$\bar{\mu}$					2″.5	2″.6	2″.7	2″.8	2″.9	3″.0					
0.7–0.9	Cal. . .	2	3	6	9	11	13	12	10	7	4	2	1			80
	Sp.-C.	0	−1	0	−1	0	−7	−7	−2	+2	−4	−2	−1			−23
	L.-C. .	−1	+2	−2	+1	−2	−10	−11	−2	−3	−3	−2	−1			−34
	$\bar{\mu}$					1″.5	1″.6	1″.6	1″.7	1″.8	1″.9	1″.9	2″.0			
0.9–1.1	Cal. . .	4	6	13	23	35	46	51	48	39	27	16	8	3	1	320
	Sp.-C.	+4	0	+4	−6	−10	−19	−42	−25	−24	−27	−16	−8	−3	−1	−173
	L.-C. .	0	+1	−1	−8	−19	−27	−42	−36	−28	−27	−14	−8	−3	−1	−213
	μ					0″.9	0″.9	1″.0	1″.0	1″.1	1″.1	1″.2	1″.2	1″.2	1″.3	

* The values of μ refer to the distances corresponding to log ρ = 0.5, 0.6, 0.8, 1.0, respectively.

The calculated number of stars of different apparent magnitudes within the sphere log $\rho = 0.5$ and the shells log $\rho = 0.5$ to 0.7, 0.7 to 0.9, etc., have been compared both with the observed data used above and with results from Luyten's list of large proper motions. The differences, observed *minus* calculated, stand opposite the headings $Sp - C$ and $L - C$, respectively, in Table VII. The last shell includes a few A stars, but in general both B and A stars are so distant that they are not involved in this comparison. The numbers from Luyten's list for the last two shells have been

[1] *Stellar Movements and the Structure of the Universe*, p. 50, 1914.

[2] *Lick Observatory Bulletin*, **11**, **1** (No. 344), 1923.

increased by 10 and 42, respectively, to allow, partially at least, for stars having motions less than his limit $\mu = 0''.5$.

The representation is perhaps all that can be expected from so detailed an analysis of a small number of stars. Nevertheless, the first two regions show the tendency revealed by Eddington's 19 stars, namely, a large percentage excess of observed stars of low luminosity.

Direct evidence on the possibility of a similar excess in the more distant regions is not easily obtained. A suggestion is afforded by the values of the mean proper motions corresponding to the various distances involved. These may be found from

$$\overline{T}_0 = 4.74\,\rho\bar{\mu}$$

where $\overline{T}_0$ is the arithmetical mean linear tangential velocity uncorrected for solar motion. Table VI of the following *Contribution* gives the numerical relation of log T_0 to M, and since log $\overline{T}_0 -$ log $\underline{T_0} = 0.121$,

$$\log \bar{\mu} = \log \underline{T_0} - (0.555 + \log \rho).$$

The values of $\bar{\mu}$ thus calculated, with $M = m + 5 - 5 \log \rho$ as argument for log T_0, are shown in Table VII. For the first two regions the probability of undiscovered proper motions larger than the mean values is very slight, but among the fainter stars there may still remain undetected a few having motions less than the mean. In the more distant regions, however, the mean proper motions are small enough to suggest that a considerable number of stars with motions larger than the mean (not to mention smaller values) are still unknown. For the region $\log \rho = 1.0$ the mean motion below $m = 12$ is about $1''.2$. In view of the statistical data on proper motions now available it seems scarcely credible that among the twenty-odd million stars between the twelfth and fifteenth magnitudes no more than the dozen assigned by the theoretical distribution should be situated within the shell in question. As far as it goes the evidence seems to indicate that complete observational data would show, here as in the nearer regions, an excess of observed over calculated numbers of stars of low luminosity.

Luyten's conclusion referred to above is perhaps correct, but his argument seems open to objection. His result depends on the

assumption that stars with proper motions exceeding a certain value can be regarded as representing those in a limited volume of space, which in general is not true. He discusses 749 stars having $\mu > 0''.5$, situated within the sphere of calculated radius $\rho = 30$ parsecs, and finds a luminosity curve very similar to that of Kapteyn and Van Rhijn. As he states, many stars of low luminosity have been omitted, because only a small fraction of those having large motions are individually known. The inclusion of these would tend to *increase* the value of the mode; but many stars of relatively high luminosity have also been omitted, because none with $\mu < 0''.5$ have been considered. The spectroscopic list alone contains about 200 such stars having distances less than 30 parsecs. Hence, for the whole sky, Luyten's number is in defect by a minimum of approximately 400 stars, systematically brighter than those he uses. Actually, the defect is much larger, for but few stars of small proper motion have yet been observed spectroscopically among the fainter magnitudes. The inclusion of all these stars would tend to *decrease* the mode of the luminosity curve, thus neutralizing more or less the increase produced by adding neglected stars of low luminosity. Moreover, it is scarcely likely that the latter, because of greater numbers, would much outweigh the influence of the brighter objects, as might be suggested by the fact that the known stars having $\mu > 0''.5$ are but a fourth of the probable total.[1] The resultant effect on the mode thus remains in doubt; but the dispersion is probably to be increased.

In conclusion, therefore, various tests agree in showing that the luminosity and density functions represent satisfactorily the data on which they are based. They also give a very fair representation of the brighter stars, which played little part in the derivation of the functions themselves. Such discrepancies as appear seem to be attributable mainly to circumstances which have only recently become known, such as the systematic difference between measured parallaxes and mean parallaxes, which perhaps is to be traced in part to the increase in the sun's velocity with increasing magnitude of the reference stars. Some revision of the constants will of course be required from time to time, to take account of the behavior of

[1] Luyten, *loc. cit.*

the sun's motion and other improvements in the data, notably more reliable counts of stars in the successive intervals of apparent magnitude which will soon be available. But as far as can be estimated, the resulting changes in the ascending branch of the luminosity function will not be large. The exact position of the mode and the character of the descending branch are uncertain, although the available data suggest that the function is asymmetrical with a relatively flat descending branch. The question is discussed from an entirely different viewpoint in *Contribution* No. 273.

Mount Wilson Observatory
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THE WIDTH OF ABSORPTION LINES IN A RAREFIED GAS

By JOHN Q. STEWART

ABSTRACT

Origin of the Fraunhofer lines.—The production of Fraunhofer lines by a cool, rarefied gas traversed by "white" radiation is discussed from the standpoint of classical electromagnetic theory. If the gas pressure is sufficiently low, dark lines are produced by scattering; so that the term "absorption line" is a misnomer. Two causes of the widening of lines are discussed: namely, thermal Doppler effect, and an increase in the number of active molecules in the line of sight. The latter is by far the more effective. Equations are developed for *line-width*, and for the minimum number of active molecules required to produce a just visible line.

Applications of the theory to the solar spectrum.—The smallness of the number of molecules required to produce a line is striking. For example, the number of sodium atoms effective in producing the *D lines in the solar spectrum* is calculated as 3×10^{16} per cm^2. Thus a very low density of gas in the solar reversing layer is indicated. The calculated weight of sodium per square centimeter column of the reversing layer is, however, considerably greater than the weight which selective radiation pressure would support.

THEORY

Although the term "absorption" often is used with reference to opacity due to scattering, one may suggest that its use be restricted to cases involving transformation of radiant to thermal energy. Scattering does not involve such transformation; the radiant energy taken up momentarily by a molecule is re-radiated without important change in frequency; and the resultant opacity is due to diffusion in direction of the incident beam of radiation. The selective opacity exhibited by very rarefied gases is, according to the quantum as well as the electromagnetic theory, caused by scattering.

The well-known classical electromagnetic theory postulates a "bound" electron in the molecule, having a "natural frequency"; such a molecule is highly effective in scattering radiation of frequency very near its natural frequency. The tuning is extremely sharp; if the impressed frequency differs from the natural frequency in the ratio, radius of electron to wave-length of the radiation, the response falls off appreciably.

The equation of motion of such a bound electron may be taken as

$$m\frac{d^2\zeta}{dt^2}-\frac{ma}{c}\frac{d^3\zeta}{dt^3}+4\pi^2 m\nu_0\zeta=eE\,, \tag{1}$$

where (employing c.g.s. and electrostatic units throughout) m is the mass, e the charge, and a the radius of a bound electron, ν_0 its natural frequency, ζ its displacement from its equilibrium position, and E the component in the direction of ζ of the alternating electric vector of the impressed radiation; while t represents time, and c the velocity of light. The relatively small forces due to the magnetic vector and to the polarization of the medium (supposed rarefied) are neglected. When the impressed radiation is monochromatic, of frequency ν and wave-length λ, $E=E_0 \cos 2\pi\nu t$. Then, if there are n scattering molecules, motionless, in unit volume, the opacity coefficient K easily is shown to be

$$K=\frac{3n\lambda^2/2\pi}{1+\lambda^2\beta^2/\pi^2a^2}, \tag{2}$$

where $\nu=\nu_0(1+\beta)$, and β is supposed small compared with unity. The gas is supposed so rarefied that $n\lambda^3$ also is small. (To take account of the randomness of molecular orientation, the right-hand side of (2) might be multiplied by a numerical factor, of the order 1/3.) If the intensity in the incident beam is I_0, the intensity in the direct beam after thickness z has been traversed is

$$I=I_0e^{-Kz},$$

where e is the base of natural logarithms.

When the Doppler effect due to the thermal agitation of the molecules is taken into account, it can be shown that (writing u for the velocity in the line of sight)

$$K=\frac{3n\lambda^2}{2\pi^{\frac{3}{2}}\alpha}\int_{-\infty}^{\infty}\frac{e^{-u^2/\alpha^2}du}{1+\lambda^2(\beta+u/c)^2\pi^2a^2}. \tag{3}$$

Here α^2 is two-thirds of the mean-square velocity of thermal agitation, or $\alpha^2=2RT/MM_0$, where R is the gas constant per molecule, T the absolute centigrade temperature, M_0 the mass of the unit of atomic weight, and M the molecular weight of the gas. Equation 3 supposes the gas so rarefied that $na\lambda^2c/\alpha$ is small.

When β is not large compared with a/c, and provided also that $ac/\lambda a$ is small, (3) leads to

$$K = K_0 e^{-c^2\beta^2/a^2}\,, \tag{4}$$

$$K_0 = \frac{\sqrt{\pi}\, ne^2}{ma\nu_0}\,, \tag{5}$$

making use of the relation

$$a = 2e^2/3mc^2\,.$$

These equations have been given by Lorentz,[1] who did not, however, note the restriction that $c\beta/a$ is not large.

When β is large compared with a/c (and also large compared with a/λ), it is readily shown from (3) that

$$K = \frac{3\pi na^2}{2\beta^2}\,, \tag{6}$$

nearly, provided $na^2\lambda/\beta^2$ is small. The corresponding index of refraction, μ, is

$$\mu = 1 - 3na\lambda^2/8\pi\beta\,.$$

Employing this value of μ, (6) can be calculated from the well-known Rayleigh formula for opacity due to scattering not too near resonance.

When opacity is due to scattering, Schuster's[2] approximation gives for the ratio, brightness of line to brightness of continuous background,

$$\psi = 1/(1 + Kz'/2)\,,$$

where z' is the thickness of gas traversed. Suppose $Kz' = k$ for the minimum observable contrast. Let $\Delta = 2\beta\lambda_0$ be the observable line width, that is, the width between the two points of minimum observable contrast. Write $N = nz'$ so that N is the number of active molecules per column of unit cross-section of gas along the line of sight.

[1] *Proc. Amsterdam Akad.*, **18**, 134, 1918.

[2] *Astrophysical Journal*, **21**, 6, 1905.

Then (5) gives for the minimum number, N', of active molecules per column of unit section required to produce an observable line, however high the dispersion,

$$N' = (8\cdot 6\times 10^5)\frac{k}{\lambda_0}\sqrt{\frac{T}{M}}, \tag{7}$$

where

$$8\cdot 6\times 10^5 = \sqrt{\frac{2R}{\pi M_0}}\frac{mc}{e^2};$$

T is centigrade absolute, and λ_0 is expressed in centimeters. This equation does not hold down to values of T so low that $ac/\lambda\alpha$ is appreciable. As the number of molecules, N, is moderately increased, (4) gives

$$\Delta = (1\cdot 3\times 10^{-6})\lambda_0\sqrt{\frac{T}{M}\log_{10}\frac{N}{N'}}, \tag{8}$$

where

$$1\cdot 3\times 10^{-6} = \sqrt{8R(\log_e 10)/M_0c^2}\,.$$

Equation 8 holds only if

$$3\pi N a^2c^2/2\alpha^2k$$

is small compared with unity. If, on the other hand, this quantity is large, (6) gives

$$\Delta = (8\cdot 2\times 10^{-13})\lambda_0\sqrt{N/k}\,, \tag{9}$$

where

$$8\cdot 2\times 10^{-13} = \sqrt{6\pi}\, a\,,$$

and the unit of length is the centimeter. Equation 9 is equivalent to

$$N = (1\cdot 5\times 10^{24})k\Delta^2/\lambda_0^2\,. \tag{10}$$

The corresponding mass of material per column of unit cross-section is

$$L = 2\cdot 5\, Mk\Delta^2/\lambda_0^2\,, \tag{11}$$

where

$$2\cdot 5 = M_0/6\pi a^2\,.$$

The standard definition of the "half-width" of an emission line affected by Doppler broadening is through the condition

$$c^2\beta^2/\alpha^2=\log_e 2$$

in (4). The "half-width" is of little significance for a strong absorption line, in relation to the observed width. It is worthy of note that (4) shows so slow a change in contrast near the center (β near 0) that practically the width of a weak line is not so small as (8) would indicate. Thus, owing to such Doppler widening, no iron line, for example, in the solar spectrum should appear narrower than about two- or three-hundredths of an angstrom.

The neglect by astrophysicists of the equations presented in this report—equations which have been more or less available for years—may have been due to lack of confidence in the assumptions involved in (1). It seems unlikely, however, that these assumptions lead to results greatly in error as regards order of magnitude. Even in early quantum theory the assumptions in question were employed, in connection with the study of black-body radiation, for calculating the energy taken up by an "oscillator."[1]

APPLICATIONS

A few of the applications of the foregoing equations are here indicated.

Equation 7 is of interest in connection with the Saha theory, when the disappearance or appearance of lines is employed as a criterion of the amount of ionization. The smallness of the number of molecules required to produce a just observable line is striking. Thus for the D lines of sodium when $T=5000°$, taking $k=2$, $\lambda=6\times10^{-5}$ cm, $M=23$, N' is given as 4×10^{11} molecules per square centimeter column. This is little greater than the number of molecules in 10^{-8} cubic centimeter of atmospheric air. Lorentz did not omit pointing out that (4) indicated very low partial pressures in the solar atmosphere of the substances producing the faintest lines. By (8), if for the D lines N were $10N'$, the resolving power necessary to observe these lines would be of the order 50,000.

[1] Richardson, *The Electron Theory of Matter*, p. 348, 1914.

As a matter of fact, each of the D lines in the solar spectrum is about 0.6 angstrom wide. Accordingly, (9) or (10) must be employed instead of (8). Equation 10, substituting $\Delta/\lambda = 10^{-4}$, $k=2$, gives 3×10^{16} as the number of the sodium atoms per square centimeter column of the sun's atmosphere effective in producing the D lines. Equation 11 gives the corresponding mass of sodium as 1.1×10^{-6} gm/sq. cm.

SELECTIVE RADIATION PRESSURE IN THE SOLAR ATMOSPHERE

Equation 11 has an immediate application to the part played by selective radiation pressure in the gravitational equilibrium of the solar atmosphere. That radiation pressure is a factor in this equilibrium has often been suggested. Eddington[1] has pointed out that, if so, the density is very small.

If I ergs per second per square centimeter are prevented by scattering in a Fraunhofer line from getting out through the solar atmosphere, the resultant upward force of radiation pressure on the scattering material is $2I/c$ dynes/sq. cm. The value of I can be calculated as a function of Δ, the line width, if the energy flux from the photosphere be assumed that from a black body at 6000° absolute. Thus

$$I = \frac{hc^2\Delta}{\lambda^5} e^{-\frac{h\nu}{RT}}, \tag{12}$$

nearly (if λ is less than about 10^{-4} cm), where h is Planck's constant, and RT is 0.52 "volts." Thus, for $\lambda = 6\times 10^{-5}$,

$$I = (8.6\times 10^9)\Delta/\lambda \text{ ergs/sec. per sq. cm.}$$

Accordingly, the radiation pressure on the sodium in a square centimeter column of the solar atmosphere due to scattering in one of the D lines, taking $\Delta/\lambda = 10^{-4}$, is about 5.8×10^{-5} dyne. If the mass of scattering sodium is 1.1×10^{-6} gm/sq. cm (as calculated above), the weight is 2.9×10^{-2} dyne, taking the acceleration of gravity at the sun's surface as 26700 cm/sec². Thus the weight of non-ionized sodium in the solar atmosphere is about five hundred times greater than the upward force of the radiation pressure corresponding to a single D line. But sodium is highly ionized

[1] *Monthly Notices of the R.A.S.*, **80**, 723, 1920.

in the solar atmosphere—perhaps only one sodium atom in one thousand is neutral; and ionized sodium, as it produces no lines of appreciable intensity in the solar spectrum, is subject to but little radiation pressure.

The same large excess of weight over available radiation pressure may be calculated for other substances in the solar atmosphere. Consequently, it may be tentatively concluded that radiation pressure plays an unimportant rôle in the gravitational equilibrium of the solar reversing layer, where most of the Fraunhofer "absorption" may be assumed to take place. In other words, the theoretical equations presented in this paper show that the slight mass of material which radiation pressure can support is insufficient to produce the observed broad Fraunhofer lines. A detailed study of conditions in the solar atmosphere is outside the purpose of the present paper.

PRINCETON UNIVERSITY OBSERVATORY

NOTE ON THE DISTANCES OF THE CLUSTER-TYPE VARIABLES

By BERTIL LINDBLAD

ABSTRACT

The effect of stream motion on proper motions and radial velocities.—The marked stream motion shown by the cluster-type variables makes it necessary, when comparing proper motions and radial velocities, to consider the apparent distribution of these stars on the sphere. It is possible that the peculiar motion of the two stars RR Lyrae and RZ Cephei may be explained by the fact that they are situated in the Milky Way near the apex of the motion of our system relative to the cluster-type variables, as it is conceivable that stars belonging to the bigger stellar system are thrown out of their courses when entering the local system. If the two mentioned exceptional stars are excluded, the material treated suggests that Shapley's distances from the period-luminosity curve are essentially correct. This result is also supported by other data for the two exceptional stars.

As remarked by Shapley,[1] the assumption of random distribution of the space velocities of the cluster-type variables when comparing proper motions and radial velocities is not permissible because of the marked streaming to be recognized in the distribution of their velocities.[2] If we assume a stream in a certain direction to be the dominating motion of a group of stars, the relations between the average radial velocity and the mean proper motion will still be the same as in the case of random distribution, provided that the stars of a certain distance are distributed uniformly on the sphere which has this distance as radius. In the case of a small number of stars, this condition may be very poorly fulfilled. If the stars are relatively few at apex and antapex, the mean of the transverse velocities will be proportionately too great compared with the mean of the radial velocities, leading to an estimation of too small distances, if formulae valid for the case of uniform distribution are used.

In the actual case of the cluster-type variables, there is another important point to be mentioned in this connection. Shapley points out that the convergent is nearly the same as that found

[1] *Harvard Circular*, No. 237, p. 4, 1922.

[2] Kapteyn and Van Rhijn, *Bulletin of the Astronomical Institutes of the Netherlands*, No. 8, 1922.

by Strömberg for stars with large space velocities. It is found to be situated in the galactic plane with the galactic co-ordinates long. 253°, lat. +1°, and not very far from the "ordinary" solar antapex. Let us for a moment accept the distances derived by Shapley from the period-luminosity curve and also the stream motion derived by him, which is 220 km per second. On account of the enormous distances of the stars in question, this "stream" motion may be assumed to be caused by the motion of our local stellar system in the greater Milky Way system.[1] The cluster-type variables are then assumed to belong to the bigger system, and have perhaps comparatively small individual velocities relative to this system. But if the cluster-type variables are distributed fairly uniformly in space, and if our local system has a motion of a couple of hundred kilometers in a certain direction relative to their centroid, it is conceivable that the variables lying *in the galactic plane* and swept over by the vast aggregation of nebulous and stellar material constituting our local system (supposed to have a galactic radius exceeding 1000 parsecs and not necessarily restricted to the local cluster of B stars) may occasionally be subject to rather violent encounters, leading to considerable changes of their course through space. It seems plausible that we shall have the best chance to witness the consequences of such encounters *near the apex point of the motion* of our system (especially if this point is situated in a region which is rich in distant stellar and nebulous matter), because in other regions of the galactic plane, i.e., deeper into the local system in the direction of the motion of the variables, these are likely to be more scarce. The stars that have suffered encounters at an early date must at the present time have drifted away very far out of our neighborhood. The effect of eventual encounters in the neighborhood of the apex point will be that the originally large radial velocities of the stars in this region of the sky will have a big chance to be diminished, and the originally small proper motions to be considerably increased. Thus, even if the cluster-type stars

[1] This view is in rough agreement with the trace of systematic motion shown in the radial velocities of ten globular clusters determined by Slipher (*Popular Astronomy*, **26**, 8, 1918). Dr. K. Lundmark has kindly communicated to me the result of an apex determination for 16 globular clusters observed by Slipher. His result for the antapex direction is long. 250°, lat. −10°, velocity 266 km.

are fairly uniformly distributed over the sky, there will be an effect tending to diminish the numerical mean of the radial velocities and to increase the mean of the proper motions. It is also clear, however, that there will be a tendency with these stars to avoid our neighborhood in space, as well as the region of the sky in the direction of the antapex (convergent of the proper motions).

It is possible that the general distribution of high-velocity vectors over one hemisphere is due to an aberration of the velocity vectors of field stars belonging to the bigger system caused by the motion of the local system toward the apex. This view is in agreement with the fact that the high-velocity vectors observed lie almost all nearly in the galactic plane,[1] but scarcely explains the wide, though one-sided, distribution of the vectors in longitude, even if the peculiar motions in the galactic plane are very great. On the other hand, if we assume a scattering of the field stars at certain points of the local system (which must reduce the mean velocity-component in the stream direction), we should expect high-velocity vectors widely distributed also in galactic latitude. But it is possible that velocity vectors of high galactic latitude will be very scarce in our neighborhood, because masses of great scattering effect (situated in the galactic plane) may be at considerable distances from our region in space, and, therefore, only those scattered stars which have vectors in the galactic plane have a chance to get within our reach as bright stars, or as stars with large proper motion.

The apex of the motion relative to the cluster-type variables is situated in the Milky Way in Cepheus. The two cluster-type variables RR Lyrae and RZ Cephei with large proper motions and comparatively small radial velocities are situated in the Milky Way not far from this apex point.

Table I presents essentially the material treated by Shapley; it is based on the proper motions derived by Kapteyn and Van Rhijn, and a few radial velocities near maximum light given by Adams[2] (for RR Lyrae the radial velocity is the "velocity of the

[1] J. H. Oort, *Bulletin of the Astronomical Institutes of the Netherlands*, No. 23, 1922.

[2] *Report of the Director, Mt. Wilson Observatory*, p. 209, 1918; *Astrophysical Journal*, **48**, 288; *Mt. Wilson Contr.*, No. 153, 10, 1917.

TABLE I

Variable	Position in 1900 R.A.	Position in 1900 Decl.	Med. Vis. Mag.	μ_α	μ_δ	$\chi-\psi$	Galactic Long.	Galactic Lat.	Distance from Apex	$v_{\text{obs.}}$	$v_{\text{comp.}}$	Radial Vel. Obs.	Radial Vel. Comp.	π (Shapley)
												km	km	
SW Androm.....	0^h18^m	+28° 51′	9.3	+0″.062	−0″.035	+ 34°	85°	−33°	34°	+0″.059	+0″.031	......	−182	0″.0012
RR Ceti........	1 27	+ 0 50	8.6	+ .011	− .068	− 16	114	−59	67	+ .066	.068	......	− 86	.0016
U Triang........	1 50	+33 17	11.6	+ .021	− .010	+ 25	106	−26	42	+ .021	.013	......	−164	.00042
SU Draconis....	11 32	+67 53	9.2	− .017	− .120	+ 3	100	+49	55	+ .121	.045	−193	−128	.0012
SW Draconis....	12 13	+70 4	10.0	....	....	......	94	+48	52	....	....	− 74	−137	.00083
RU Can. Ven....	13 55	+32 7	11.1	− .022	− .008	− 44	19	+73	81	+ .017	.024	......	− 34	.00052
W Can. Ven.....	14 2	+38 18	10.3	− .039	− .015	− 42	37	+70	75	+ .031	:032	......	− 58	.00072
ST Virginis......	14 23	− 0 27	10.8	+ .008	− .016	+ 56	315	+53	106	+ .010	.027	......	+ 62	.00060
RS Bootis.......	14 29	+32 12	10.3	+ .011	− .008	+ 83	18	+66	78	+ .002	.034	− 51	− 47	.00076
SW Aquarii.....	21 10	− 0 20	10.4	− .053	− .039	− 42	20	−33	59	+ .049	.029	......	−112	.00072
SX Aquarii.....	21 31	+ 2 47	11.8	+ .005	− .027	+ 19	26	−35	56	+ .026	.014	......	−125	.00036
SU Aurigae.....	4 50	+30 24	8.7	+ .064	− .060	+ 11	140	− 7	68	+ .086	.069	......	− 82	.0016
ST Ophiuchi....	17 29	− 1 1	11.6	− .008	+ .005	+270	350	+15	83	.000	.019	......	− 28	.00042
RR Lyrae.......	19 22	+42 36	7.2	− .107	− .215	+ 21	43	+12	32	+ .224	.074	− 69	−186	.0030
XZ Cygni.......	19 30	+56 10	9.7	....	....	......	56	+16	24	....	....	−196	−201	.0010
RZ Cephei......	22 36	+64 20	10.1	+ .082	+ .179	− 17	77	+ 6	7	+ .188	.005	− 3	−218	.00085

system" according to Kiess,[1] and for RZ Cephei the mean of five determinations at different phases according to Luyten).[2] The antapex direction was assumed according to Shapley's determination (R.A. $= 157°$, Decl. $= -57°$). The first part of the table gives the stars situated outside the Milky Way, the second part the stars in the Milky Way.

The first columns of the table are self-explanatory. The seventh column gives the angle between the antapex direction and the direction of the observed proper motion. The eleventh and twelfth columns give the observed motion in the antapex direction and the same motion computed on the basis of Shapley's parallaxes and the stream velocity 220 km. The next two columns give the observed and computed radial velocities. The median visual magnitude is according to Shapley; the galactic co-ordinates from *Harvard Annals*, **56**, Table VIII, for the stars contained in that table.

Turning our attention to the first part of the table, the stars outside the Milky Way, we find in the mean,

$$\bar{\upsilon}_{\text{obs.}} = +0\overset{''}{.}040, \qquad \bar{\upsilon}_{\text{comp.}} = 0\overset{''}{.}032, \text{ (10 stars)},$$

Radial velocity obs. $= -106$ km,

Radial velocity comp. $= -104$ km (3 stars).

The radial velocities of the remaining stars of this group range between $+62$ km and -182 km. The average computed radial velocity for this group is 103 km.

In the Milky Way group the first two stars with large apex-distances possibly behave normally. Including these in the foregoing group, we get the following mean values of υ:

$$\upsilon_{\text{comp.}} < 0\overset{''}{.}030, \quad \bar{\upsilon}_{\text{obs.}} = +0\overset{''}{.}021, \quad \bar{\upsilon}_{\text{comp.}} = 0\overset{''}{.}021, \text{ (6 stars)},$$
$$\upsilon_{\text{comp.}} > 0\overset{''}{.}030, \quad \bar{\upsilon}_{\text{obs.}} = +0\overset{''}{.}061, \quad \bar{\upsilon}_{\text{comp.}} = 0\overset{''}{.}047, \text{ (6 stars)}.$$

Among the three others, which have small apex distances, XZ Cygni has exactly the demanded radial velocity, but Tucker has derived the proper motion $0\overset{''}{.}1$, which does not agree with the computed "parallactic" motion.[3] RR Lyrae and RZ Cephei are

[1] *Lick Observatory Bulletin*, **7**, 146, 1913.

[2] *Publications of the Astronomical Society of the Pacific*, **35**, 68, 1923.

[3] *Lick Observatory Bulletin*, **9**, 145, 1918.

quite exceptional in their motion, with too small radial velocities and too large proper motions.

The facts mentioned here indicate that a rough statistical comparison between radial velocity and proper motion must not be made in the case of these stars; it is necessary to consider more in detail the organization of the group of stars in question. As far as the rather meager material published up to the present time is concerned, the proper motions and radial velocities indicate that Shapley's parallaxes deduced from the period-luminosity curve are correct within 25 per cent.

That the two stars diverging from the general stream, RR Lyrae and RZ Cephei, are high luminosity stars comparable with those belonging to the stream is indicated by evidence of another kind. For RR Lyrae we have the following parallax determinations:

$+0.006 \pm 0.006$	trigonometric,	Mt. Wilson
$+0.0005 \pm 0.0046$	trigonometric,	Neubabelsberg
0.004	spectroscopic,	Mt. Wilson
0.003	period luminosity,	Mt. Wilson

For RZ Cephei an inspection of objective prism spectra taken by the present writer at Upsala[1] indicates that the star at maximum light ought to be placed in a certain type of class A, which is characterized by spectra with hydrogen lines decidedly less winged than, for instance, in the spectrum of Sirius, though considerably more winged than in the spectrum of α Cygni. The parallactic motion for 23 stars of this character within a limited area of the sky indicates an absolute magnitude of -0.5, in good accordance with Shapley's value for RZ Cephei deduced from the length of the period.

NOTE

1. R. E. Wilson[2] has given new determinations of those proper motions in Table I which rest on data from meridian observations. The changes of importance for our discussion are the following:

	μ_α	μ_δ	$v_{obs.}$
SW Andromedae	$-0''.009$	$-0''.004$	$0''.000$
SU Draconis	$-\ .006$	$-\ .131$	$+\ .130$
SU Aurigae	$+\ .025$	$-\ .034$	$+\ .042$

[1] *Harvard Bulletin*, No. 778, 1922. [2] *Astronomical Journal*, **35**, 26, 1923.

With these new values we get for the mean observed and computed values of v for the stars outside the Milky Way

$$\bar{v}_{\text{obs.}} = +0\overset{\prime\prime}{.}035, \qquad \bar{v}_{\text{comp.}} = 0\overset{\prime\prime}{.}032 \text{ (10 stars)},$$

and for the group with $v_{\text{comp}} > 0\overset{\prime\prime}{.}030$ (excluding the last three stars of Table I),

$$\bar{v}_{\text{obs.}} = +0\overset{\prime\prime}{.}045, \qquad \bar{v}_{\text{comp.}} = 0\overset{\prime\prime}{.}047 \text{ (6 stars)}.$$

The average residual for twelve stars is $0\overset{\prime\prime}{.}022$.

Wilson also gives proper motions for the stars RX Eridani and RW Cor. Bor., but with large probable errors, and further for R Muscae, which has the relatively long period 0.882 days (spectrum G5). The insensible proper motion found for this star agrees with the position near the antapex, but might also indicate that the star belongs to the local system.

2. If we suppose that the cluster-type variables in the Milky Way with motions deviating from the general stream follow roughly hyperbolic orbits with major axes of about 200 parsecs (as sharp encounters between single stars must be extremely scarce, the orbits are probably of large dimensions, governed by the gravitational forces of vast aggregations of matter), and with the velocity 220 km in infinity, the disturbing mass ought to be of the order of magnitude 10^9 solar masses. As this must be a very considerable part of the total mass of the local system,[1] it is probable, if the foregoing reasoning is correct, that we have to consider the galactic clouds in Cepheus and Cygnus as belonging to this system, and those clouds as a whole as contributing to the disturbing gravitational force.

The possible analogy between the motion of the cluster-type variables and that of high-velocity stars in general further suggests the following contribution to an explanation of Kapteyn's two star-streams. The characteristic distribution of high-velocity vectors in our neighborhood of space is taken to be a consequence of the scattering of the vectors by the gravitational forces of the local system, in addition to the aberration of the vectors by

[1] Kapteyn gives for the number of stars in the "stellar system" 47.4×10^9, *Astrophysical Journal*, **55**, 309, 1922; *Mt. Wilson Contr.*, No. 230.

the motion of this system toward the apex. In this way we get the two opposite streams of widely scattered stars going nearly transverse to the antapex direction, in addition to the group of stars still moving nearly in this direction.[1] But on account of the possibly great dispersion in the velocity-distribution of the field stars (which must be considered originally under the attraction of the Milky Way system as a whole), there will also belong organically to these groups stars with somewhat smaller velocities relative to our system, sufficiently small so that the stars will remain as members of the local system. It is therefore probable that the local system will show the same directions of preferential motion as the system of high-velocity stars. The stars in the group with motions toward the antapex will tend to reduce the speed of the local system when absorbed as members, and will also cause a preferential motion in the apex-antapex direction, whereas stars from the groups with nearly transverse velocities will unite in causing a direction of preferential motion at right angles to the apex-antapex direction. It may therefore not be a mere chance that the apex of the motion of the local system is almost exactly at right angles to the line combining the true vertices of the two-star streams.

OBSERVATORY, UPSALA
June 1923

[1] G. Strömberg, *Mt. Wilson Communications,* No. 79, 1922; *Mt. Wilson Contr.*, No. 245; *Astrophysical Journal*, **56**, 265, 1922. J. H. Oort, *loc. cit.*

THE 3883 CYANOGEN BAND IN THE SOLAR SPECTRUM[1]

By RAYMOND T. BIRGE

ABSTRACT

CN lines in the solar spectrum.—A critical study of the 3883 band, as it appears in the solar spectrum, has yielded a list of 180 lines, apparently not blended, and suitable for an investigation of the Einstein shift. Spectrograms from the Mount Wilson Observatory and numerous enlargements have been used in this work. The lines are weighted from 1–16, very low to very high weight.

Review of previous work.—A recomputation of previous work on the Einstein shift, by St. John, Grebe, and Bachem, using the present weighting of the lines, leads essentially to the original results, so that the disagreement between the results of these investigators still remains. But many of the lines of highest weight, according to this investigation, have not as yet been employed.

Relative λ in arc and in sun.—The relative wave-lengths of the 180 chosen lines in the arc and in the sun have been measured on a Mount Wilson plate of 2.5 mm per angstrom dispersion, and are found to be the same within limits of error. Many previously measured discrepancies are found to be due to the use of undetected blends. The present result indicates that any solar shift, if present, is common to all the lines of this band.

Table and plates.—The 180 lines, as well as 135 other lines in the region λ 3881 to λ 3773, which seemed worthy of mention, are listed in a table, which contains in addition to the wave-lengths all other available information. The same lines are indicated on plates, which show the sun and arc spectra in coincidence.

The 3883 band, whose chemical origin is still in dispute, has been studied more than any other, and it took on added importance when Einstein predicted a shift to the red in the case of solar lines, amounting in this portion of the spectrum to about 0.008 A, for the lines of this band are known to be unaffected by pressure, an influence which produces a serious objection to the use of the lines of most so-called "line spectra." Moreover, these lines are quite uniformly sharp and symmetrical, all observed dissymmetries being seemingly produced by the overlapping of two or more series lines.

In 1917 St. John[2] used 43 line-groups of this band (calling a doublet, whether resolved or unresolved, a line-group) and obtained an average shift to the red of only about 0.001 A. In 1919 Grebe and Bachem[3] used 36 lines and obtained an average shift of 0.004 A.

[1] Presented to the American Physical Society, April, 1923 (*Physical Review*, **21**, 712, 1923).

[2] *Astrophysical Journal*, **46**, 249, 1917.

[3] *Verhandlungen der deutschen physikalischen Gesellschaft*, **21**, 454, 1919.

In 1920 these authors[1] ran photometric curves for the 36 lines and found apparently only 11 really valid for use, the others being blends in the sun. These 11 lines gave a shift of 0.005 A.

In 1921 Grebe[2] used 100 solar lines assigned by Rowland to carbon (including all evident blends), and obtained the theoretical shift by the use of an indirect process. This consisted of computing the difference between the Rowland[3] and the Uhler and Patterson[4] wave-lengths of 10 lines for which the various investigators on the spectral shift were in fair agreement, and then correcting this difference by the mean observed shift to obtain the absolute difference between the Rowland and I.A. systems at the points in question. A smooth curve through these corrected differences then gave such a comparison over the interval λ 3870 to λ 3856, and values from this curve were then compared with the difference between the actual Rowland and Uhler and Patterson readings to obtain the shift for 100 lines in this interval. The validity of the results thus depends wholly on the accuracy of the observed shift for the few lines on which the method is based. As is indicated in the table,[5] only 2 of the 10 lines are considered of high weight, and it thus seems as if relatively little weight could be attributed to the final results.

Then King[6] showed that the relative intensity of the lines of this band varied with the temperature, and the author[7] showed that the variation was that to be expected on the quantum theory of band spectra. But before this, Grebe and Bachem's 11 so-called "good lines" had been investigated,[8] and only 2 found which were not, in the arc, complexes of two or more series, so that any change of relative intensity in the solar spectrum, compared to the arc, would be likely to yield false results.

But now it appears[9] that the distribution of intensity of the 3883 *CN* solar lines is the same as that in a carbon arc run on 3–4 amperes and 160 volts—an intensity distribution indicating a source at 4000° C. By chance, both St. John and King used the

[1] *Zeitschrift für Physik*, **1**, 51, 1920.

[2] *Ibid.*, **4**, 105, 1921.

[3] *Astrophysical Journal*, **1**, 29, 1895.

[4] *Ibid.*, **42**, 434, 1915.

[5] See note b of table.

[6] *Astrophysical Journal*, **53**, 161, 1921.

[7] *Ibid.*, **55**, 273, 1922. To be called B. 1.

[8] Birge, *Science*, **53**, 368, 1921. To be called B. 2.

[9] B. 1, p. 289.

carbon arc under these conditions. So, by being careful to use such an arc, one may employ lines that are actually complexes of two or more series, provided the resulting complex is fairly symmetrical and easy to measure with accuracy, in the sun and in the arc, since the distribution of intensity will be the same in both cases.

This enormously increases the amount of available material. Accordingly, a systematic study has been made of the entire solar region in which the lines of this band occur with sufficient intensity for measurement. This region includes about 35 per cent of the whole band, and extends from λ 3881 to λ 3773. In it Rowland[1] lists about 900 solar lines, and Uhler and Patterson[2] list 864 *CN* arc lines.

In the present work a number of St. John's original negatives have been used, including one in the fourth order of the 30-foot grating, having a dispersion of 2.45 mm to the angstrom. Plates by King, with this band in emission in the arc and in the furnace, and in absorption in the furnace, have been used as well. An intensive study has been made to obtain solar *CN* lines which were not blends. This is more difficult than it might at first appear, for the observed intensity of an absorption line is influenced by all of the lines in its vicinity, a very strong line even several angstroms distant producing a distinct effect.[3] This is not true for emission lines. Thus a solar *CN* line frequently appears stronger than expected from the arc intensities, but it is almost impossible to decide whether all of the enhanced intensity is due to neighboring strong lines, or whether at least a part is due to a blended line. Also, it is almost impossible to compare black lines with white lines, as to relative intensity, and a much better comparison can be obtained by using a projected negative in contact with a printed enlargement of the same size, for the projected negative is itself a negative, and thus a black arc line (projected) can be compared with a black solar line (print). Gradations of intensity are indicated far better with black lines than with white lines. Printed enlargements of King's absorption plate were also used, in compari-

[1] *Astrophysical Journal*, **1**, 29, 1895.

[2] *Ibid.*, **42**, 434, 1915.

[3] See note 7 of table.

son with solar enlargements, except in the region near the head of the band, for, outside of that region, the relative intensities of the various series lines are practically independent of temperature; and hence the fact that the distribution of intensities in the absorption plate indicates a temperature of about 3000° C., while that of the solar plate indicates 4000° C., does not introduce any error. This comparison was by far the most sensitive used.

The final result of this work has been a list of 189 lines, numbered consecutively in the table, and labeled on the plates, weighted from 16–1 (very high 16, high 8, medium 4, low 2, very low 1). All of the lines of this band are probably doublets, the doublet separation near the zero intensity point being zero.[1] In the case of resolved doublets, only cases where both members of the doublet appear unblended in the sun were used,[2] for St. John and Ware[3] have found that the measured separation of a close doublet is dependent on various factors, and systematic differences are very likely to appear between the measured separation of emission and of absorption doublets. The author found this to be true in the measurements to be mentioned.

The best lines, in my opinion, are five doublets of intensity 0 or 1 on Rowland's scale, the majority of them lying to the violet of the region used by previous investigators of the Einstein shift. Nearer the head of the band there is so much overlapping of series, as is evident from the plates, that the choice of suitable lines is very difficult, and many of the recorded weights are subject to much uncertainty.

One of the noticeable features of the previous work on the Einstein shift has been the discordant character of the results. The present study indicates that in a number of cases of very large observed shifts, either to the red or to the violet, the line considered is actually a blend, in the sun, thus accounting for the anomalous result. A recomputation of all previous work, using the new weighting indicated in the table, leads, however, to no appreciable change in the results. Thus, of Grebe's 100 lines only the numbered lines 32–72 of the table seem entitled to any weight, but these give

[1] See p. 280 of B. 1.

[2] But see notes q and 21 of the table. [3] *Astrophysical Journal*, 44, 15, 1916.

the same average result as the 100, showing that, as expected, the chance effect of blends cancels out. Of Grebe and Bachem's 36 lines, 26 (and not merely 11) seem entitled to some weight,[1] but the weighted average shift agrees exactly with the original value of 0.004 A, obtained from all 36 lines. St. John's results, as recomputed, give a shift of 0.0016 A by his method (*a*), and 0.0010 A by his method (*b*), agreeing approximately with the original results.

A question that naturally arises is whether the *relative* wave-lengths of the lines of this band are the same in the sun as in the arc. If they are, then, if any of the lines of the band are definitely proved to show a real shift in the sun, it will indicate that all of the lines of the band are similarly affected. In order to test this point, St. John's fourth-order plate, previously mentioned, was used for the measurement of all the lines of this band which had previously been selected as entitled to some weight.[2] This list included about 80 lines in addition to the 189 numbered lines. Most of these 80 lines are listed in the table, with the reason for dropping them; in general, because, although clearly visible on the printed enlargements, they were found too faint for accurate measurement in the comparator. Aside from the dropping of these lines, no change of weighting has been made as a result of the measurements. The *relative* wave-lengths were measured on exposures of the arc and of the sun, both exposures appearing on the same photographic plate, but somewhat separated so that only one could be measured at a time, with accuracy. The *difference* in the relative wave-lengths appears in column 7 of the table.

A preliminary test of a number of selected standards indicated that the dispersion of both the arc and of the solar exposures was the same, within limits of error of setting. But the measurement of the entire list of lines occupied several days, and during this time there was a slight change of dispersion, due mainly to change of humidity. This was checked, as usual, by setting on various fiducial specks on the plate. The relative wave-lengths given in

[1] See p. 289 of B. 1 for further remarks on these lines.

[2] The measurements were made on a 20 cm comparator, for which the author is indebted to the Rumford Committee of the American Academy of Arts and Sciences.

TABLE I

1	2	3	4	5	6	7	8	9
1	3880.791	0.931	A_2 (16) i	C (1)			− 20	1
2	.731		A_2 (16) i		4	− 2		2
3	.670	0.815	A_1 (16) i	C (1)			+ 30	
		0.684		C (000)				R″
	.392	0.532	A_4 (17) i	C (2)			− 20	R′
		0.328		−C (0)				p
4	.092	0.235	A_2 (18) i	C (0)	8	+ 7	+ 10	
5	.033	0.175	A_2 (18) i	C (0)			± 0	
6	79.964	0.105	A_1 (18) i	C (1)			− 10	S″
		9.986		C (∞)				R″
7	.712	9.851	A_2 (19) i	C (0)	8	− 5	− 30	
8	.654	9.796	A_2 (19) i	C (0)			± 0	3
9	.578	9.716	A_1 (19) i	C (1)			− 40	G′, S′
		9.578		C (000)				R″
10	.311	9.458	A_2 (20) i	C (0)	4	± 0	+ 50	4
11	.253	9.394	A_2 (20) i	C (0)			− 10	
		9.178		C? (∞N)				5
12	78.891	9.037	A_2 (21) i	C (0)	2	+40	+ 40	
13	.828	8.975	A_2 (21) i	C (0)			+ 50	
	.577	8.720	F*	Fe (7Nd?)				6
14	77.832	7.972	A_1 (23) m	C (0)	1	−26	− 20	
15	.506	7.646	A_2 (24) i	C (0)	3	+27	− 20	7
16	.446	7.587	A_2 (24) i	C (0)			− 10	
	.351	7.481	A_1 (24) w	C (1)			−120	G′, S, b, 8
	.005	7.121	A_2 (25) i	C (4Nd?)			+ 70	9
	76.938		A_2 (25) i					
17	.481	6.622	A_2 (26) i	C (0)	8	− 7	− 10	7
18	.415	6.556	A_2 (26) i	C (0)			− 10	
	.315	6.448	A_1 (26) i	C (0)			− 90	G′, S, 10
19	75.939	6.083	A_2 (27) i	C (0)	4	+41	+ 20	e, 11
20	.873	6.019	A_2 (27) i	C (0)			+ 40	e, 11
		5.220		V (2)				12
21	74.190	4.328	A_2 (30) i	C− (0)	8	− 6	− 40	R
22	.123	4.258	A_2 (30) i	C (00)			− 70	13
		4.091		Co–C (4)				p
23	73.501	3.636	A_2 (31) i	C (0)	1	− 8	− 70	q
24	.371	3.504	A_1 (31) w	C− (0)	2	− 4	− 90	G″, R‴, 14
25	72.724	2.859	A_1 (32) i	C (1Nd?)	1	−32	− 70	15
26	.180	2.312	A_2 (33) i	C (0)	1	−25	−100	S, q
	71.566	1.693	A_2 (34) i	(0)			−150	R′
	.501		A_1 (34) i					16
27	.133	1.259	B_1 (5) m	C (0)	1	+52	−160	
28	.010	1.145	B_1 (6) m	C (0)	2	+ 8	− 70	
29	70.876	1.018	A_2 (35) B_1 (7) is	C (1)	4	+72	± 0	G′, e, 17
	.808	0.932	A_2 (35) w	C (0)			−180	R′
30	.719	0.848	B_1 (8) m	C (00)	2	+20	−130	
31	.665	0.797	A_1 (35) B_2 (8) ms	C (0)			−100	
32	.550	0.685	B_1 (9) B_2 (9) m	C (0N)	1	+104	− 70	G, 17
33	.358	0.493	B_1 (10) B_2 (10) m	C?	1	+64	− 70	R, 17
34	.283	0.405	B_2 (10) f	C?	4	−16	−200	R, 18
35	.069	0.204	A_2 (36) B_2 (11) is	C (0)	4	± 0	− 70	
36	69.831	9.960	B_2 (12) f	C (00)	1	−36	−130	
		9.745		C (1)				R′

TABLE I—*Continued*

1	2	3	4	5	6	7	8	9
......	....	9.444	....	C (oNd?)	...	......	......	R′
......	3869.066	9.179	B_1 (15) w	C (o)	...	......	− 60	w
......	.021		B_2 (15) w					w
......	68.829	8.941	B_1 (16) w	C (o)	...	......	−300	w
......	.721	8.873	B_2 (16) w	C (1)	...	......	+100	w
37...	.645	8.785	A_2 (38) f	C (oo)	4	− 4	− 20	S″, 19
38...	.571	8.700	A_2 (38) B_1 (17) ms	C (o)	4	−12	−130	G′, S, b, 19
39...	.487	8.625	B_2 (17) f	C (oo)	4	− 1	− 40	S″, 19
......	.407	8.539	A_1 (38) B_2 (17) ms	C (1)	...	......	−100	G′, S″, R′
......	....	8.372	....	−C (o)	...	......	......	p
40...	.124	8.261	B_1 (18) w	C (o)	2	+44	− 50	G′
41...	.033	8.171	B_2 (18) f	C (oo)	2	+68	− 40	....
......	....	8.060	....	Fe–C (2)	...	......	......	p
......	67.860	7.996	A_2 (39) f	C? (ooo)	...	......	− 60	R, f′
......	.779	7.906	A_2 (39) B_1 (19) ms	C– (o)	...	......	−150	b
......	....	7.791	....	C (o)	...	......	......	20
42...	.619	7.758	A_1 (39) B_2 (19) is	C–V (1)	2	− 8	− 30	R‴
43...	.384	7.520	B_1 (20) f	C (o)	1	+23	− 60	....
44...	.302	7.449	B_2 (20) f	C (o)			+ 50	....
45...	.062	7.205	A_2 (40) f	C (o)	4	+12	+ 10	G′, S, b
46...	66.982	7.118	A_2 (40) B_1 (21) ms	C (o)	8	−18	− 60	G′, S
47...	.555	6.692	B_1 (22) f	C (o)	4	−16	− 50	....
......	.396	6.526	B_2 (22) w	C (oo)	...	......	−120	z
48...	.240	6.380	A_2 (41) w	C (oo)	2	−20	− 20	21
49...	.168	6.306	A_2 (41) w	C (oo)	2	+ 6	− 40	21
......	.108	6.238	B_1 (23) f	C (oo)	...	......	−120	f″
50...	65.991	6.122	A_1 (41) B_2 (23) is	C? (3Nd?)	1.5	−26	−110	S″, G′, 22
......	....	6.046	....	C? (ooo)	...	......	......	R″
......	.648	5.793	B_1 (24) f	C (o)	...	......	+ 30	f′
......	.399	5.554	A_2 (42) w	C (o)	...	......	+130	z
......	.327	5.454	A_2 (42) w	C (o)	...	......	−150	z
51...	.151	5.282	A_1 (42) B_1 (25) is	C? (3)	2	+ 5	−110	S″, G′, R, b
......	.085	5.213	B_2 (25) w	C (ooo)	...	......	−140	f′
......	.010	5.134	B_2 (25) w	C (o)	...	......	−180	R′, G′
52...	64.667	4.802	B_1 (26) f	C (oo)	1	+20	− 70	....
53...	.300	4.438	A_1 (43) i	C (3)	8	+ 6	− 40	S, G″, b
......	63.993	4.113	B_2 (27) w	C (o)	...	......	−220	R′
54...	.593	3.734	A_2 (44) B_1 (28) ws	C (oo)	2	+ 2	− 10	....
55...	.528	3.655	B_2 (28) w	C (oo)			−150	....
56...	.390	3.533	A_1 (44) B_2 (28) ibc	C (3N)	2	+60	+ 10	S″, G″
57...	62.976	3.113	B_2 (29) w	C (oo)	8	− 6	− 50	....
58...	.900	3.041	B_2 (29) w	C (oo)			− 10	....
59...	.768	2.897	A_2 (45) w	C (oo)	8	−16	−130	....
60...	.694	2.827	A_2 (45) w	C (ooo)			−90	23
61...	.489	2.627	A_1 (45) B_1 (30) ibs	C?–(2)	4	± 0	− 40	S″, G′, b, 24
62...	.403	2.541	B_2 (30) w	C (ooo)	2	−36	− 40	....
63...	.324	2.458	B_2 (30) w	C (oo)			− 80	G′
64...	61.711	1.847	B_2 (31) C_2? Ib	C (2N)	2	+38	− 60	G″, 25
......	.567	1.734	A_1 (46) IbdsH$_2$	C (2)	...	......	+250	w
......	.543	1.681	f	C (2)	...	......	− 40	w, 26
......	.456	1.592	C_1 C_2 f	C (oo)	...	......	− 60	f′
......	60.916	1.067	A_2 (47) f	C (oo)	...	......	+ 90	R′
65...	.829	0.963	C_1 (9) C_2 (9) mb	C (o)	4	− 4	− 80	....

TABLE I—*Continued*

1	2	3	4	5	6	7	8	9
......	....	0.863	....	C (000)	...	......	......	R″
66...	3860.497	0.630	B_2 (33) C_2 (11) m	C (0)	2	−72	− 90	....
67...	.424	0.566	B_1 (33) C_1 (11) m	C (0)			± 0	....
......	.221	0.354	C_1 (12) C_2 (12) ms	C? (0N)	...	......	− 90	R, z
......	59.783	9.876	B_2 (34) f	C (00)	...	......	−490	R′
......	.424	9.568	C_1 (15) C_2 (15) wb	C? (0)	...	......	+20	R, f′
......	.277	9.415	B_1 (35) wc	C (000)	...	......	− 40	f′
......	.204	9.355	B_2 (35) w	Fe (3)	...	......	+ 90	27
......	.115	9.246	B_2 (35) C_1 (16) w	C? (00)	...	......	−110	R′
68...	58.993	9.128	A_2 (49) w	C (00)	1	+50	− 70	....
......	.918	9.052	A_2 (49) C_2 (17) m	C? (1)	...	......	−80	R′
......	.836	9.000	C_2 (17) f	(1N)	...	......	+220	R′
69...	.683	8.822	A_1 (49) id	C (2N)	8	− 4	− 30	S′G″, b, 28
70...	.591	8.722	B_1 (36) C_2 (18) m	C (0)	4	−40	−110	S′, G′, e
......	.515	8.642	B_2 (36) w	C (000)	...	......	−150	f′
......	.450	8.606	B_2 (36) C_1 (18) w	C (0)	...	......	+140	R′
71...	57.994	8.146	A_2 (50) w	C? (00)	2	−14	+100	R, 29
72...	.896	8.033	B_1 (37) mc	C? (1)	4	+ 5	− 50	R, G′, G
73...	.814	7.955	B_2 (37) w	C? (0)	2	+ 8	− 10	R
74...	.449	7.580	C_2 (21) w	C (00)	2	−76	−110	q
75...	.334	7.473	C_1 (21) wb	C (0)	4	+12	− 30	....
76...	.158	7.288	B_1 (38) C_2 (22) mc	C?- (1)	4	−25	−120	R‴, S
......	.074	7.215	B_2 (38) C_1 (22) f, bl	C? (000)	...	......	− 10	f
......	56.994	7.135	B_2 (38) A_2 (51) f	C? (000)	...	......	− 10	f′
77...	.922	7.063	A_2 (51) C_1 (22) m	C? (0)	8	−21	− 10	R, G″, S
78...	.658	6.802	A_1 (51) id	C? (2N)	4	+ 9	+ 20	R, S″
......	.234	6.367	B_2 (39) w	(0)	...	......	− 90	R′
......	55.964	6.107	A_2 (52) w	(00)	...	......	+ 10	R′
......	.622	5.766	A_1 (52) B_1 (40) Is	C (3)	...	......	+ 20	z
......	....	5.721	....	Fe–C (1)	...	......	......	p
......	.123	5.259	C_1 (26) w	C? (00)	...	......	− 60	R, f
79...	54.932	5.088	A_2 (53) w	C (00)	2	+16	+140	....
80...	.851	4.989	B_1 (41) A_2 (53) ms	C (1)	4	−35	− 40	S″, e
......	.744	4.869	H_4	C (0)	...	......	−170	30
81...	.662	4.808	w	C (0)	1	−16	+ 40	....
82...	.566	4.707	A_1 (53) IbdsH_4	C (2Nd?)	2	± 0	− 10	G′
......	.252	4.401	f	C (00)	...	......	+ 70	R′
......	.199	4.343	f	C (0)	...	......	+ 20	R′
83...	.062	4.191	B_1 (42) C_1 (28) m	C (0)	8	−52	−130	S, G′, e
......	53.909	4.040	w	C (00)	...	......	−110	f′
			blurr					
......	.812	3.967	w	C (000)	...	......	+130	f′
......	.686	3.805	f	C (00)	...	......	−230	R′
......	.487	3.620	A_1 (54) id	C (2d?)	...	......	− 90	R′, S″, G′, b
84...	52.908	3.045	w	C (00)	2	+28	− 50	....
......	.774	2.899	A_2 (55) f	C (000)	...	......	−170	R?, w′
85...	.706	2.845	A_2 (55) m	C (00)	2	+36	− 30	....
86...	.402	2.541	A_1 (55) B_1 (44) Ids	C? (2Nd?)	2	+80	− 30	S″, G″, 31
......	.121	2.245	w	C (00)	...	......	−180	R?, z
87...	51.681	1.815	A_2 (56) mb	C (00)	2	+64	− 80	....
88...	.592	1.733	A_2 (56) m	C (00)	1	+28	− 10	....
89...	.527	1.672	B_1 (45) m	C (0)	4	+24	+ 30	....
90...	.285	1.427	A_1 (56) C_1 (33) id	C (2Nd?)	8	+14	± 0	S′, G″
91...	.168	1.306	w	(00)	4	+ 6	− 40	R
92...	.082	1.220	f	(000)			− 40	R
93...	50.647	0.781	B_1 (46) C_1 (34) m	C? (0)	8	−14	− 80	R, S″

TABLE I—*Continued*

1	2	3	4	5	6	7	8	9
......	3850.525	0.700	A_1 (57) mb	C (oo)	...	......	− 40	R
......		0.626	A_1 (57) mb	C (o)	...	......		R′
94...	.405	0.536	w	(oo)	3	−28	−110	....
95...	.302	0.440	w	(oo)	2	+22	− 40	....
96...	.158	0.300	A_1 (57) id	C (1Nd?)	2	+12	± 0	G′
......	49.876	0.013	w	(o)	...	......	− 50	R, f′
97...	.749	9.886	B′ (47) mb	C (oN)	2	+27	− 50	....
......	.260	9.400	f	C (ooo)	...	......	− 20	f′
98...	.008	9.140	A_1 (58) id	La–C (3d?)	2	−22	−100	32
99...	48.844	8.979	B_1 (48) mb	C (1N)	4	+16	− 70	S, G″
......	.697	8.840	C_1 (37) f	(oo)	...	......	+ 10	R, f″
......	.612	8.745	w	(oo)	...	......	− 90	R′?, f″
......	.536	8.667	f	(oo)	...	......	−110	R′
100...	.194	8.329	A_1 (59) w	C (oo)	1	−80	− 70	....
101...	.104	8.249	f	(oo)			+ 30	R
101′...	.052	8.186	f	(oo)			− 80	R, 33
......	47.269	7.394	C_1 (39)	C (ooN)	...	......	−170	R′
......	46.965	7.087	B_1 (50) mc	C (1)	...	......	−200	R′
102...	.677	6.814	A_1 (60) i	C (1)	4	+ 3	− 50	S″, G′
103...	.633	6.777	A_1 (60) i	C (1)			+ 20	
104...	.537	6.666	C_1 (40) f	(oo)	2	−32	−130	R, S
......	.000	6.131	B_1 (51) m	C (2)	...	......	−110	R′, S
105...	45.832	5.949	A_1 (61) f	C (ooNd?)	2	−16	− 70	....
	.797		C_1 (41) f					
106...	.320	5.461	m	C (ooo)	1	+10	− 10	....
107...	.018	5.149	B_1 (52) C_1 (42) is	C (1)	4	−10	−110	S, G′, b
108...	44.250	4.378	A_1 (62) C_1 (43) i	C (4d?)	1	+84	+ 80	S″, G″
	.206		A_1 (62)					
......	43.831	3.962	f	(ooo)	...	......	−110	R, f
109...	.457	3.600	C_1 (44) mb	C (oN)	4	− 2	+ 10	....
......	.009	3.127	A_1 (63) B_1 (54) Is	C (3)	...	......	− 60	R′
......	42.973		A_1 (63) B_1 (54) Is					
110...	.647	2.779	C_1 (45) w	C (o)	4	− 1	−100	S, G′
......	.464	2.587	f	C (oo)	...	......	−190	f′
111...	41.953	2.082	B_1 (55) md	C (o)	2	−54	−130	....
......	.816	1.959	C_1 (46)	C (o)	...	......	......	f′, 29
112...	.753	1.862	A_1 (64) i	C (2d?)	2	−10	−110	....
	.710		A_1 (64) i					
113...	40.105	0.239	C_1 (48) w	C? (o)	1.5	+47	− 80	R
......	38.759	8.888	B_1 (58) m	C (1Nd?)	...	......	+ 70	z
......	.719		B_1 (58) m					
......	....	8.435	....	Mg–C (25)	...	......	......	p
......	37.877	8.035	A_1 (67) i	C (o)	...	......	+160	z
......	.825	7.961	A_1 (67) i	C (o)	...	......	− 60	z
114...	.651	7.768	B_1 (59) m	C (1d?)	3	−12	− 60	....
	.614		B_1 (59) m					
115...	.420	7.559	C_1 (51) w	C (ONd?)	2	+ 5	− 30	....
116...	36.540	6.689	A_1 (68) B_1 (60) C_1 (52) I	C (1)	2	+ 8	+ 70	S″
117...	.494	6.639	A_1 (68) B_1 (60) C_1 (52) I	C (1)			+ 30	
......	....	6.337	....	C (ooo)	...	......	......	R″

TABLE I—*Continued*

1	2	3	4	5	6	7	8	9
......	....	5.862	....	−C (oood)	...	......	......	p
......	3835.549	5.689	C_1 (53) w	C (oNd?)	...	......	− 20	f′
118...	.389 .343	5.509	B_1 (61) m B_1 (61) D_1ms	C (1d?)	2	+ 2	+ 10	....
119...	.202	5.342	A_1 (69) i	C (o)	3	+12	− 20	....
120...	.147	5.298	A_1 (69) i	C (o)			+ 90	....
......	34.635	4.762	f	C (ooo)	...	......	−150	w′
......	.573	4.709	C_1 (54) D_1ws	C (o)	...	......	− 60	w′, 34
......	33.950	4.096	f	(ooo)	...	......	+ 40	R, z
121...	.611	3.744	C_1 (55) m	C (o)	4	−32	−90	S′, 35
122...	32.639	2.790	C_1 (56) m	(o)	1	+18	+ 90	R
......	31.187	1.334	D_1 fb	(ooNd?)	...	......	+ 50	R, w
123...	.064 .005	1.174	A_1 (72) i A_1 (72) i	C− (3d)	4	+18	− 25	R‴, S″
124...	30.665	0.801	B_1 (65) m	C (o)	4	+ 7	− 60	S″
125...	.611	0.745	B_1 (65) m	C (o)			− 80	
......	.366	0.513	C_1 (58) D_1ws	(o)	...	......	+ 50	R, w′
126...	28.211	8.360	A_1 (74) B_1 (67) Is	C (o)	3	+ 7	+ 70	36
127...	.155	8.296	A_1 (74) B_1 (67) Is	Ti–C (1)			− 10	
......	27.372	7.519	C_1 (61) w	C (o)	...	......	+ 50	f′
128...	26.770	6.905	A_1 (75) D_1ic	C (o)	4	+ 7	− 70	....
129...	.708	6.843	A_1 (75) $D_1$is	C (oo)			− 70	....
......	.307	6.449	C_1 (62) w	C? (o)	...	......	± 0	R, z
......	.190	6.343	C_1 (62) w	C? (o)	...	......	+110	R. z
......	25.674	5.820	B_1 (69) D_1ms	C? (1N)	...	......	+ 40	R, z
......	.616	5.736	B_1 (69) m	(1N)	...	......	−220	R, z
130...	.307	5.448	A_1 (76) i	C (o)	4	− 5	− 10	....
131...	.239	5.373	A_1 (76) C_1 (63) ic	C (o)			− 80	....
132...	.125	5.256	C_1 (63)	(oo)	2	−11	−110	R, S
......	24.795	4.936	D_1f	(o)	...	......	− 10	R, f′
......	.739	4.887	D_1f	(oo)	...	......	+ 60	R, f
133...	23.823	3.953	A_1 (77) $D_1$is	C (o)	4	− 6	−120	....
134...	.760	3.893	A_1 (77) $D_1$is	C (o)			− 90	....
135...	.083	3.228	B_1 (71) m	C (o)	8	− 4	+ 30	....
136...	.024	3.163	B_1 (71) m	C (o)			− 30	....
......	22.645	2.785	w	(oo)	...	......	− 20	R, f″
137...	.322	2.470	A_1 (78) i	C (o)	16	+ 6	+ 60	S″
138...	.259	2.406	A_1 (78) i	C (o)			+ 50	
139...	19.278	9.412	A_1 (80) i	C (1)	16	+ 3	− 80	S″
140...	.213	9.346	A_1 (80) i	C (1)			− 90	
141...	.063	9.197	B_1 (74) id	C (1Nd?)	4	+ 8	− 80	S
......	18.739	8.891	D_1f	C? (ooo)	...	......	+100	R
......	.670	8.759	D_1w	C (1)	...	......	−530	R′
......	.080	8.225	fb	(ooo)	...	......	+ 30	R, f
142...	17.384	7.523	B_1 (75) m	−C (o)	6	+10	− 30	R‴, 37
......	.142	7.290	f	C (ooo)	...	......	+ 60	f
143...	.055	7.198	C_1 (70) w	C (oo)	2	+26	+ 10	38
......	16.985	7.114	C_1 (70) w	(ooo)	...	......	−130	R, z
......	.622	6.779	D_1 f	C (ooN)	...	......	+150	f′
......	.547	....	D_1 f	....	...	......	......	f′
......	.170	6.332	A_1 (82) B_1 (76) ic	C? (o)	...	......	+200	R, z
......	.101	6.252	A_1 (82) i	C? (oN)	...	......	+ 90	R, z

TABLE I—*Continued*

1	2	3	4	5	6	7	8	9
......	3815.236	5.352	f	(ooN)	...	......	−260	f
......	.079	5.222	f	(ooN)	...	......	+ 10	f
......	14.333	4.500	D_1f	C (oo)	...	......	+250	R′
......	13.331	3.537	C_1 (73) D_1ms	C (2)	...	......	+640	R′
144...	12.062	2.205	B_1 (79) C_1 (74) is	C (o)	16	+ 2	+ 10	39
145...	11.985	2.126	B_1 (79) C_1 (74) is	C (o)			− 10	39
146...	10.615	0.761	B_1 (80) m	C (o)	16	± 0	+ 40	....
147...	.542	0.681	B_1 (80) m	C (o)			− 30	....
148...	09.755	9.894	A_1 (86) i	C (o)	16	− 4	− 30	....
149...	.690	9.834	A_1 (86) i	Mn–C (o)			+ 20	R‴, 40
......	08.637	8.770	f	(ooo)	...	......	+ 10	R, f
150...	06.440	6.586	A_1 (88) $D_1$is	C (o)	8	− 6	+ 40	....
151...	.377	6.511	A_1 (88) $D_1$is	C (o)			− 80	....
152...	05.524	5.669	C_1 (79) m	C (oo)	1	−10	+ 30	....
153...	.449	5.589	C_1 (79) m	C (oo)			− 20	....
......	.191	5.337	D_1w	C (oo)	...	......	+ 40	f″
......	.128	5.256	D_1w	C (oo)	...	......	−140	f″
......	04.696	4.836	A_1 (89) B_1 (84) Is	C (1)	...	......	−20	41
154...	.176	4.317	C_1 (80) w	C (oo)	1	+ 5	− 10	....
155...	.098	4.237	C_1 (80) w	C (oo)			− 30	....
......	03.928	4.044	D_1f	C (oo)	...	......	−260	R′
......	.680	3.816	f	(oo)	...	......	− 60	R, f
......	.570	3.711	w	(oo)	...	......	− 10	R, f
......	.090	3.228	A_1 (90) B_1 (85) Is	C (1)	...	......	− 40	42
......	.013	3.141	A_1 (90) i	C (o)	...	......	−140	R′
156...	02.808	2.952	C_1 (81) D_1ms	C (o)	2	+48	+ 20	43
157...	.727	2.870	C_1 (81) D_1ms	C (o)			+ 10	43
......	.577	2.725	w	(ooo)	...	......	+ 60	R, f″
......	.486	2.614	w	(oo)	...	......	−140	R′
......	01.375	1.511	A_1 (91) C_1 (82) is	C (1)	...	......	− 60	R′
158...	00.311	0.457	E m	C? (oN)	1	− 6	+ 40	R
159...	.098	0.256	B_1 (87) m	C (o)	2	+12	+160	....
160...	.032	0.174	B_1 (87) C_1 (83) is	C (o)			± 0	....
161...	3796.184	6.329	A_1 (94) i	C (o)	8	+17	+ 30	....
162...	.104	6.247	A_1 (94) i	C (o)			+ 10	....
......	95.805	5.952	C_1 (86) E ms	C (oo)	...	......	+ 50	f
......	.722	5.880	C_1 (86) E ms	C (oo)	...	......	+160	f
163...	93.787	3.921	B_1 (91) is	C (oo)	3	− 3	− 80	....
164...	.699	3.846	B_1 (91) is	C (ooo)			+ 50	....
165...	92.639	2.788	A_1 (96) i	C (o)	2	−14	+ 70	....
166...	.565	2.702	A_1 (96) cdi	C (o)			− 50	....
......	91.085	1.246	E ws	C (o)	...	......	+190	R′
167...	89.040	9.186	A_1 (98) i	C (o)	8	− 5	+ 40	....
168...	88.968	9.110	(A_1 98) i	C (o)			± 0	....
169...	85.574	5.719	B_1 (96) m	C (oo)	4	+18	+ 30	....
170...	.495	5.641	B_1 (96) m	C (oo)			+ 40	....
171...	.390	5.539	A_1 (100) C_1 (93) is	C (o)	4	+ 8	+ 70	....
172...	.319	5.457	A_1 (100) C_1 (93) is	C (o)			− 40	....
173...	83.905	4.035	B_1 (97) C_1 (94) E Is	C (oo)	8	+ 8	−120	....
174...	.819	3.954	B_1 (97) C_1 (94) E Is	C (oo)			− 70	....
......	.093	3.224	D_1′w	(ooo)	...	......	−110	R, f

TABLE I—*Continued*

1	2	3	4	5	6	7	8	9
175...	3780.509	0.654	B_1 (99) mc	C (00)	2	−20	+ 30	....
176...	.434	0.564	B_1 (99) mc	C (00)			−120	....
......	....	0.223	....	C (000)	...	......	....	R″
177...	79.806	9.989	A_1 (103) i	C (0)	4	−26	+410	44
178...	.732	9.871	A_1 (103) i	C (0)			− 30	....
......	78.808	8.939	B_1 (100) D_1′is	C (1)	...	......	−110	R′
179...	77.919	8.075	A_1 (104) i	C (0)	2	+36	+140	....
180...	.842	7.982	A_1 (104) i	C (0)			− 20	....
181...	.755	7.897	C_1 (98) w	C (00)	1	+ 4	± 0	....
182...	.664	7.809	C_1 (98) w	C (00)			+ 30	....
183...	75.287	5.431	B_1 (102) m	C (00)	4	+14	+ 20	....
184...	.204	5.342	B_1 (102) m	C (00)			− 40	....
185...	74.107	4.247	A_1 (106) i	C (00)	2	+36	− 20	....
186...	.030	4.170	A_1 (106) i	C (00)			− 20	....
187...	73.556	3.695	B_1 (103) E′ ms	C (00)	1	+11	− 30	....
188...	.475	3.609	B_1 (103) m	C (00)			− 80	45

EXPLANATION OF NOTES, COLUMN 9 OF TABLE I

b One of Grebe's ten basic lines, used for obtaining the absolute difference between the Rowland and the I.A. system.

e Column 7 outside limits of experimental error.

f Too faint for use in arc and in sun.

f′ Too faint for use in sun.

f″ Too faint for use in arc.

G The 76 lines listed in the table, from 32 to 72, form a portion of Grebe's 100 lines. The other 24 are indicated by Rowland as blends, and are omitted.

G′ One of Grebe and Bachem's 36 lines.

G″ One of Grebe and Bachem's 11 good lines, selected from the 36.

p Purely a foreign line, not a *CN* blend.

q Should not have been included as only one line of pair, in the arc.

R A *CN* line.

R′ A *CN* blend.

R″ Not a *CN* line.

R‴ A *CN* line. not a *CN* blend.

S Used by St. John, with high weight.

S′ Used by St. John, with medium weight.

S″ Used by St. John, with low weight.

w Too poor for use, in arc and in sun.

w′ Too poor for use, in sun.

z Too close to adjacent strong line for use.

1. With the exception of 3881.628 (R) and 3881.346 (R), which are not *CN* lines, Rowland's designations from here to the head of the band seem to be correct, but the component lines are too close together for accurate measurement.

2. This line is accurately measurable on St. John's solar plate.

3. The triplet composed of lines 7, 8, and 9 ($m' = 19$) furnishes the most sensitive test of the solar temperature of *CN*, as mentioned on page 289 of B. 1. The three lines are of equal intensity, in both sun and arc, on the published plate, indicating the equality of temperature and justifying the use of such an arc, in measuring the spectral shift. Note that Rowland does not give equal intensities. His intensities would indicate a still lower solar temperature.

4. A_1 is now far enough from A_2 so that the A_2 doublet can be used separately. This line is too close to the *Cr–C* line 3879.331 (R) to be given high weight.

5. This line is not listed by Uhler and Patterson. It appears on King's arc spectrograms, and is especially strong on King's furnace and absorption plates. Origin unknown.

6. Listed by Uhler and Patterson as probably a foreign line. It is undoubtedly the *Fe* line 3878.576.

7. The doublets 15, 16 and 17, 18 furnish a good example of the effect of neighboring heavy lines in an absorption spectrum. These two doublets differ in intensity, in the arc, by only 4 per cent, according to the author's intensity distribution curve, but in the sun 15, 16 appears fully twice as strong, due to the *Fe* (8) line at 3878.152 (R).

8. Rowland's intensities for the m′=24 triplet group indicate that this line is a blend. But on St. John's solar plates it has the proper intensity, and should have been included in the list with medium weight.

9. This line is evidently a blend, but not nearly as strong as Rowland's published intensity makes it. A comparison of the plates published in this article with Rowland's published intensities will indicate many apparent discrepancies, a fact that has already been noted by St. John (*op. cit.*, p. 255).

10. Should have been included and given low weight, due to faintness in arc.

11. These lines *may* be contaminated, in the sun, with the vanadium 3875.897 line.

12. This line appears quite strongly in King's furnace-absorption spectrogram of *CN*.

13. Lines 21 and 22 are better for accurate measurement, on the negative, than the published prints indicate. They are of equal intensity.

14. One of the two good lines, selected by the author (B. 2) from Grebe and Bachem's eleven good lines. Given low weight here because one component of a close doublet in the sun. See note 28.

15. This line is non-symmetrical, being a mixture of A_1 and a perturbed A_2 line. The theoretical position of A_1 is 3872.722. The recorded I.A. reading is from King's furnace plate, where A_1 is much stronger than the contaminating A_2 line. Uhler and Patterson give 3872.739.

16. This line is on St. John's solar plate, although omitted by Rowland, but too poor to use.

17. The three lines 29, 32, and 33, giving apparent solar shifts to the red, of the magnitude of the Einstein shift, are typical of the possibilities of error in this work. The weighting of all lines was made before taking the actual readings, and has not been changed, but a re-examination of these three lines, with arc and solar exposures simultaneously under the cross-hair of the comparator, revealed the following facts: Line 29 is definitely wider in the sun than in the arc, and is evidently blended in the sun with a line to the red. Lines 32 and 33 are of the same width, in sun and arc, but each is non-symmetrical to the red, in the arc. The setting was made on the center of gravity, which thus lay to the violet of the geometric center. But in the sun the non-symmetrical character could not be detected, and the setting was made on the actual center of the relatively broad line, this setting thus being to the red of that in the arc. In all three cases the relative shift can be detected in the comparator by mere inspection.

18. Rowland's measurement evidently incorrect.

19. On the published plate, the arrows for 37, 38, and 39 should each be shifted one line to the left, making 38 a strong line, and 37 and 39 the weaker ones.

20. This is not *CN*, but vanadium 3867.642, indicated by Rowland as being a blend with 42, which in turn is purely a *CN* line. But because of this close component to the red, 42 is given low weight.

21. Lines 48, 49 should have been used as a doublet of weight 4.

22. May be a *CN* blend. In any case, non-symmetrical in sun and arc, and so poor. The lower arrow for this line, on the plate, is misplaced to the left.

23. Solar intensity is (00), not (000).

24. Certainly not a *CN* blend. The theoretical positions of A_1 and B_1 are 3862.488 and 3862.458, respectively, but B_1 is very weak, being the first line to the violet of the missing line m′=29.

25. Probably H_3; i.e., the turning-point of the C series.

26. Arc intensity should be I, not f.

27. A *CN* blend, but the *Fe* line probably 100 times as strong.

28. The other of the author's two good lines. See note 14.

29. Arc λ measured by author.

30. Nothing definite to set on. Edge of band.

31. Arc λ by author, agreeing with St. John, but disagreeing with Uhler and Patterson by 0.011 A. This line, showing, according to St. John as well as the author, a large shift to the red, is very questionable. The relative strength and position of the line, using Grebe and Bachem's microphotometric curves and all other available material, indicate that in the arc it is a blend with a fairly strong line of unknown origin at 3852.391 (Uhler and Patterson's λ), while the A_1 line lies at .397, its theoretical position. In the sun, on the other hand, there is evidence of an additional blend, on the red, causing the apparent shift.

32. *La* very faint, if present.

33. From here on, *CN* lines in the sun, unblended but evidently unsuitable for use, are in general not listed. But all cases where Rowland's designation seems incorrect are listed.

34. Listed by Rowland 3834.609. Evidently a typographical error.

35. Much stronger than adjacent C_1 lines, and solar shift too large. May be blend, in arc.

36. *Ti* is at λ 3828.174, and is probably weakly present in sun. Setting was made on center of doublet, as this is unaffected.

37. B_1 perturbation. Other B_1 line at 3817.840.
38. Center line of a triplet, therefore usable.
39. Recorded λ agrees with theoretical position of B_1, while C_1 is theoretically 0.005 A to the red.
40. *Mn* line, by Rowland, but not found by others. See Volume V, page 748, of Kayser's *Handbuch*.
41. Overlooked when measurements were made. Should be included, with high weight.
42. Slightly wider in sun than in arc, and probably a weak blend.
43. C_1 and D_1 not coincident.
44. Rowland's λ must be incorrect. Separation of pair normal on St. John's plate.
45. All further *CN* pairs to λ 3760 are blended, and from there on are too faint.

the table are therefore the results obtained by correcting for this shift, the maximum correction thus applied being 0.0025 A. For closely adjacent lines the published differences are in all cases a direct representation of the original comparator readings, and it is only for lines at some distance apart that the correction exceeds 0.001 A.

DESCRIPTION OF TABLE

Column 1 of the table gives the numbering of all lines considered suitable for use in measuring the spectral shift. These lines are marked with arrows on Plates I and II, and similarly numbered. Each solar line is given a separate number, but in a great many cases, as mentioned, the two or three lines of a close doublet or triplet are to be used together. Separate settings were made on each line, in such cases, and the mean setting has been used. Frequently, settings were made as well on the center of the doublet. These checked very closely with the mean setting just mentioned. As indicated in the notes, the lines λ 3877.351, λ 3876.315, and λ 3804.696 should have been included among the numbered lines, while the lines numbered 23, 26, 29, 32, 33, 74, 86, and 121 probably should have been omitted from the numbered list.

Column 2 gives the wave-length in I.A., as measured by Uhler and Patterson[1] with a few noted additions or corrections by the author.

Column 3 gives the wave-lengths as measured by Rowland.[2]

Column 4 gives the series designation, intensity, and general appearance, according to Uhler and Patterson, with the following additions by the writer: (*a*) The values of m′ for the B_1, C_1, A_2, B_2, and C_2 series, where $m'=0$ for $\Delta\nu=0$ approximately; i.e., the older method of numbering the lines of a band. (*b*) The identification of all C_2 lines, of the C_1 lines for m′ less than 47, and of the A_2 lines for

[1] *Astrophysical Journal*, **42**, 434, 1915. [2] *Ibid.*, **1**, 29, 1895.

PLATE I

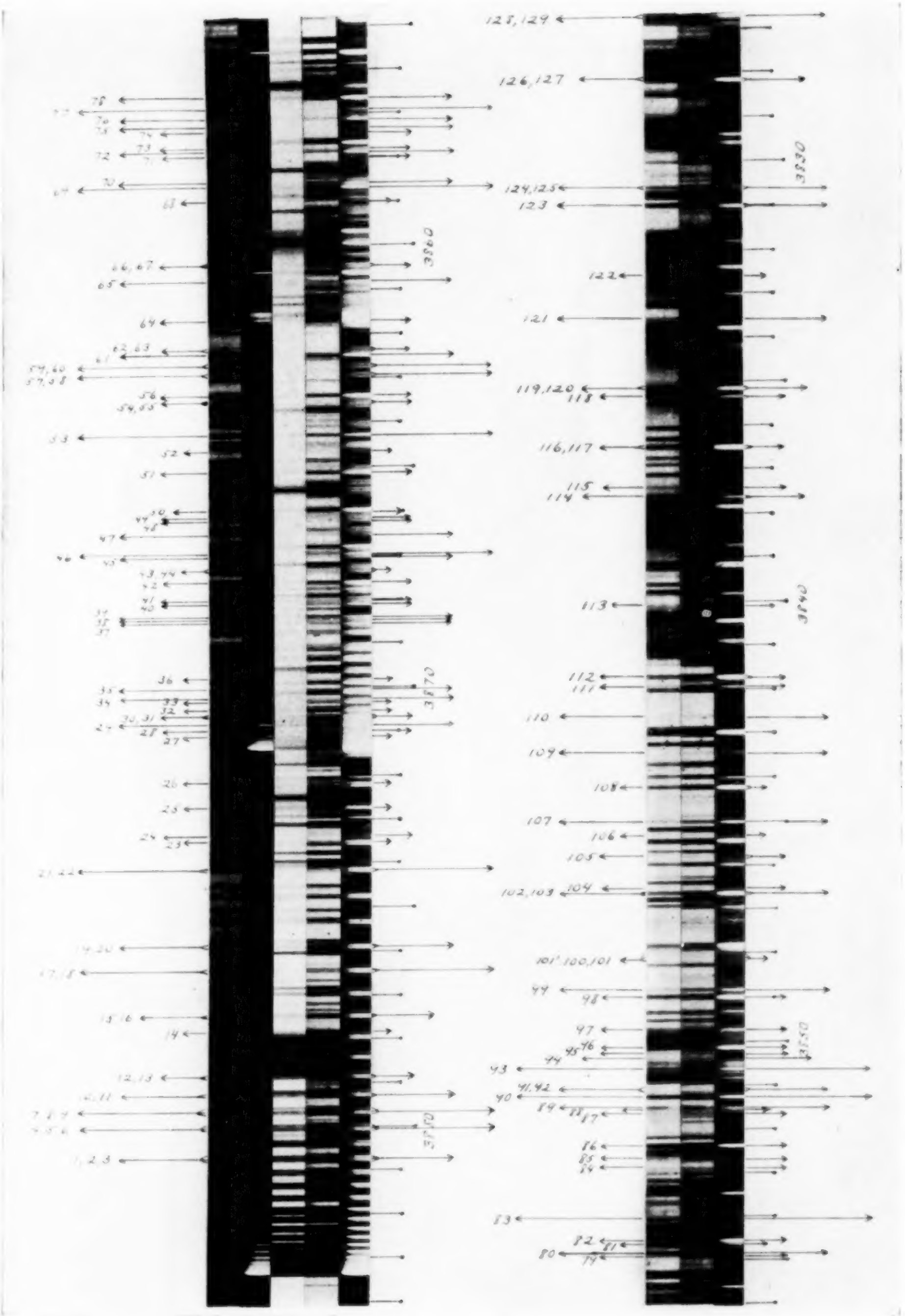
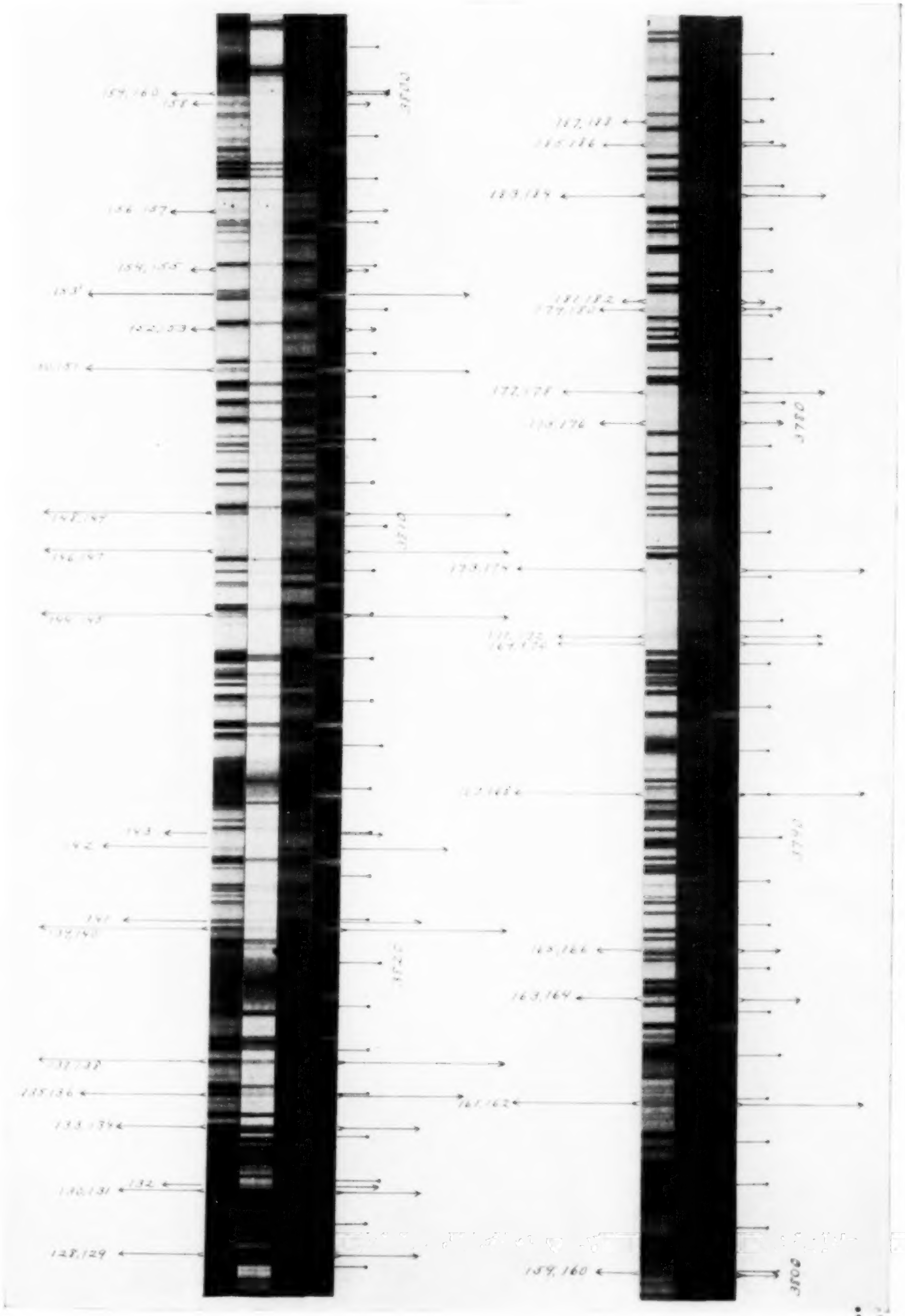


PLATE II



$m'=46$–61. (The identification has been carried to $m'=64$ at λ 3842.252 and .182.) It should be noted that the intensity designations (i = intense, m = medium, w = weak, f = faint, F = very faint) refer only to the *relative* intensity of adjacent lines, so that an "intense" line near the head of the band is, according to the writer's intensity curves (B. 1), several hundred times as intense as a similarly labeled line near the end of the band; but Rowland's intensity designations, given in column 5, refer to the relative intensity over a long range.

In column 5, besides Rowland's intensity, is given his assignment (C indicates a *CN* line, C– or –C indicates a *CN* blend.)

Column 6 gives the relative weighting by the author, from 16 (very high) to 1(very low).

Column 7 gives the relative wave-lengths (solar minus arc) in ten-thousandths of an angstrom, as measured by the author.

Column 8 gives the relative wave-lengths in the same unit, using the figures of columns 2 and 3, with column 3 lowered by 0.142 A, the average difference between the two systems of wave-lengths over most of the region used. This average difference seems to be about 0.142 A at λ 3885, to drop rapidly to 0.137 A (the figure used by St. John) at λ 3870, to remain constant at that value to λ 3840, and then to rise to 0.142 A at λ 3815, and from there to remain constant to λ 3770.

Column 9 lists the notes. Remarks referring to one line only are numbered, those used repeatedly are lettered. The wave-lengths are in I.A. unless specified as (R).

PLATES

Plates I and II, from enlarged prints of arc and solar spectrum, made from the same original spectrogram (the fourth-order Mount Wilson plate) but with differing exposures, bring out the various lines. The exposure across the plates, from left to right, is not quite uniform, being greater on the left edge. The wave-length scale is in I.A., and is accurate to about 0.03 A. The prints are so arranged that in each case the lower arrows are in contact with an arc spectrum, while the upper arrows indicate the corresponding solar lines. When two or more lines are used as a group, a single arrow is drawn, the doublet or triplet tail indicating the individual

lines. The length of the arrows indicates the relative weighting of the lines.

RESULTS

The experimental results of this article are contained in column 7 of the table. They may be summarized as follows: Five very high-weight doublets (line-groups) give an average difference in the relative wave-lengths of arc and solar lines of 0.00028 A. Eighteen high-weight groups average 0.00122 A, or only 0.00098 A with line 83 omitted. Thirty-seven medium-weight groups average 0.00143 A, or 0.00102 A with four groups omitted. (One of the four, line 29, is discussed in note 17. Another group, lines 19 and 20, is discussed in note 11. The other groups are lines 70 and 80.) Forty-three low-weight groups average 0.00272 A, and twenty-two very-low-weight groups average 0.00336 A. In addition there are nine groups of intermediate weights.

With the exception of the five line-groups mentioned, the discrepancies in relative wave-length are in all cases within the limits of experimental error, and the conclusion, therefore, seems justified that the relative wave-lengths of the 3883 *CN* lines are the same in the sun as in the arc, and that previously measured discrepancies are due to disturbing factors of one kind or another. Because of the relatively large number of lines seemingly available, the list here published should furnish material for a conclusive test of the spectral shift predicted by Einstein's theory. In the case of ordinary line series there is no accurate criterion for comparing the relative intensities in the arc and in the sun for the purpose of detecting blends. Most of the lines are quite strong and so subject to considerable error of measurement, because there is only a narrow range of suitable exposure time for the most accurate measurement of a spectral line. With such enormous ranges of intensity as are found in the case of line spectra, there are likely to be on any given plate only a very few lines of suitable intensity. This is not true of band lines, and this fact, with the other favorable properties of band lines, seems to make them greatly superior to other lines in investigations on the Einstein shift.

PHYSICAL LABORATORY
UNIVERSITY OF CALIFORNIA
August 1923

ON THE SPECTRUM OF NEUTRAL HELIUM. II

By C. V. RAMAN AND A. S. GANESAN

ABSTRACT

A rejoinder to Dr. Silberstein's reply.—The two points raised by Dr. Silberstein in reply to the criticisms of his combination formula are answered. The figures have been recalculated, taking the maximum limit of the quantum numbers to be that proposed by Dr. Silberstein himself, and the number of fortuitous coincidences to be expected between the observed lines of the helium spectrum and those given by the formula is calculated and found to agree fairly well with the actual number. The view that the coincidences noticed are due to mere chance is maintained.

In his rejoinder[1] to our criticism of his paper[2] on this subject Dr. Silberstein has attempted to dispose of our contention that the approximate coincidences between some of the lines of the helium spectrum and those given by his combination formula are purely fortuitous. He raises two points in his rejoinder. The first is that we took the limits of the quantum numbers to be

$$3 \leqq n \leqq 9 \text{ and } 4 \leqq m \leqq 32 \tag{1}$$

and thus obtained a high estimate of the probability of chance coincidences. To find whether this reply meets our objection, we have recalculated our figures, confining ourselves to the limits

$$3 \leqq n \leqq 8 \text{ and } 4 \leqq m \leqq 20 \tag{2}$$

now proposed by Dr. Silberstein himself as suitable for a test of the correctness of his views. Further to clinch the matter, a calculation has been made, as explained below, of the number of fortuitous coincidences to be expected.

The first step in the work is to make a table of all the lines given by the combination formula, satisfying the conditions (2) and lying in the frequency interval

$$19800 < \gamma < 37800. \tag{3}$$

We find there are 760 lines, not 631 as per Silberstein. It is found further that the lines are not distributed uniformly throughout

[1] *Astrophysical Journal*, **57**, 240, 1923.

[2] *Ibid.*, **56**, 119, 1922.

the whole interval of frequency. There are numerous small gaps and a small number of relatively large gaps. Thus one-third the number of gaps is less than 10, about two-thirds the number are less than 25, while only 10 per cent of the gaps exceed 50. Within the frequency limits given by (3), which are separated by a gap of 18,000, there are 96 observed lines of the helium spectrum. To determine how many chance coincidences may be expected between these and the 760 lines given by the formula, we may imagine the latter to be represented by points on a straight line, and that 96 shots are fired at random so as to hit the line. If there are N gaps of average interval x, the expectation of the number of shots that will hit some of these gaps is evidently

$$\frac{Nx}{18000} \times 96.$$

Further, if any shot falls within a frequency gap x, the maximum distance between it and the nearest point on the straight line is $x/2$. Actually all distances between 0 and $x/2$ are equally probable, and if y be the number of shots that may be expected to hit the gap x, we may divide them further into n groups of y/n shots each, each group hitting the line at distances of $\frac{x}{2(n+1)}, \frac{2x}{2(n+1)}, \ldots \frac{nx}{2(n+1)}$ from the nearest point. In practice it is sufficient to take n moderately large, say 5. In this way, by taking the different gaps in the line, it is possible to work out the number of shots that may be expected to hit the line within a specified distance from the nearest point. The argument may be illustrated by the following example. The number of frequency gaps between 20 and 25 is 83. The aggregate gap interval is 1925. The number of lines that may be expected to fall within this gap is

$$\frac{1925}{18000} \times 96 = 10.$$

Of these 10 lines we can expect 2 to fit the nearest one with an error of 1.9, 2 with an error of 3.8, 2 with 5.7, 2 with 7.6, and 2 with an error of 9.5. Calculating in this way, we have constructed the

following table, giving the relations between the permissible error and the fits calculated and actual.

TABLE I

Permissible Error	Fortuitous Fits		Permissible Error	Fortuitous Fits		Permissible Error	Fortuitous Fits	
	Calculated	Actual		Calculated	Actual		Calculated	Actual
1	3	7	11	61	65	21	85	87
2	9	15	12	65	69	22	87	87
3	17	24	13	68	70	23	88	87
4	27	33	14	72	73	24	89	88
5	33	42	15	73	74	25	90	89
6	43	47	16	77	76	26	92	89
7	49	49	17	79	79	27	93	90
8	54	51	18	81	80	28	93	92
9	56	55	19	82	83	29	95	92
10	60	63	20	83	87	30	96	93

The table shows that except for small errors less than 3, the agreement between the calculated and the actual number of fits is nearly perfect. Hence, even confining ourselves to the conditions given in (2), we find that the coincidences are accidental, and our original contention still holds.

Silberstein says that "44 lines coincide with observed lines whose total is 96. The mean deviation is $|\delta\nu| = 2 \cdot 5.7$. Now a straightforward computation will show that the probability of such an event considered as a fortuitous set of coincidences is . . . . well below 10^{-13}." It is a matter for regret that Dr. Silberstein does not explain clearly in his reply to our criticism what his "straightforward method of computation" is. In view of the figures given above, it would seem that there must be some fundamental error which vitiates his method of calculation. Perhaps he has overlooked the fact that the theoretical lines according to his formula are not uniformly distributed, but fall into groups, and that this must profoundly affect the probability of random coincidences. But if his formula claims to explain the spectrum of helium, why should 44 frequencies alone coincide and not all the 96?

CALCUTTA
August 22, 1923

GENERAL INDEX TO *ASTROPHYSICAL JOURNAL*, VOLS. XXVI–L (1907–1919)

The Editors announce with pleasure that a General Index to Volumes XXVI–L (1907–1919) of the *Astrophysical Journal* has been prepared by Professor Storrs B. Barrett, of the Yerkes Observatory, and is now ready for distribution.

This General Index is uniform in style with the General Index to Volumes I–XXV (1895–1907), which was also compiled by Mr. Barrett and appeared in 1908. It is arranged by subjects and by authors, forming a volume of 116 pages, and is of the same format as this *Journal*.

From our own experience in the use of the first General Index, we can speak with assurance of the value of this second General Index to all users of the *Journal*.

The University of Chicago Press has fixed the price of the new index at \$2.50, in paper covers, and makes a special offer, until July 1, 1924, of the two General Indices for \$3.25.

To subscribers whose set of the *Journal* does not include the early volumes, the first General Index will serve as a bibliography of the papers appearing in those volumes, many of which are now out of print.